Advanced Higher Mathematics

Time: 3 hours

NATIONAL QUALIFICATIONS

Specimen Question Paper

Read carefully

1. Calculators may be used in this paper.
2. There are five Sections in this paper.

   Section A assesses the compulsory units Mathematics 1 and 2
   Section B assesses the optional unit Mathematics 3
   Section C assesses the optional unit Statistics 1
   Section D assesses the optional unit Numerical Analysis 1
   Section E assesses the optional unit Mechanics 1.

   Candidates must attempt **all** questions in Section A (Mathematics 1 and 2) **and one** of the following:

   Section B (Mathematics 3)
   Section C (Statistics 1)
   Section D (Numerical Analysis 1)
   Section E (Mechanics 1).
3. Full credit will be given only where the solution contains appropriate working.

Section A (Mathematics 1 and 2)

All candidates should attempt this Section.

Marks

A1. (*a*) Find partial fractions for

$$\frac{4}{x^2-4}.$$ **2**

(*b*) By using (*a*) obtain

$$\int \frac{x^2}{x^2-4}\,dx.$$ **4**

A2. The performance of a prototype surface-to-air missile was measured on a horizontal test bed at the firing range and it was found that, until its fuel was exhausted, its acceleration (measured in $\mathrm{m\,s^{-2}}$) t seconds after firing was given by

$$a = 8 + 10t - \frac{3}{4}t^2.$$

(*a*) Obtain a formula for its speed, t seconds after firing. **2**

(*b*) The missile contained enough fuel for 10 seconds. What horizontal distance would it have covered on the firing range when its fuel was exhausted? **3**

A3. Use the substitution $x = 4\sin t$ to evaluate the definite integral

$$\int_0^2 \frac{x+1}{\sqrt{16-x^2}}\,dx.$$ **5**

A4. Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= -1\cdot 1 \\ x + 3y + 2z &= 0\cdot 9. \end{aligned}$$ **5**

A5. (*a*) Find the derivative of y with respect to x, where y is defined as an implicit function of x by the equation

$$x^2 + xy + y^2 = 1.$$ **2**

(*b*) A curve is defined by the parametric equations

$$x = 2t + 1, \qquad y = 2t(t-1).$$

(i) Find $\dfrac{dy}{dx}$ in terms of t. **2**

(ii) Eliminate t to find y in terms of x. **1**

Marks

A6. Let u_1, u_2, ..., u_n, ... be an arithmetic sequence and v_1, v_2, ... , v_n, ... be a geometric sequence. The first terms u_1 and v_1 are both equal to 45, and the third terms u_3 and v_3 are both equal to 5.

(*a*) Find u_{11}. **2**

(*b*) Given that v_1, v_2, ... is a sequence of **positive** numbers, calculate $\sum_{n=1}^{\infty} v_n$. **3**

A7. Use induction to prove that

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

for all positive integers n. **5**

A8. Let the function f be given by

$$f(x) = \frac{2x^3 - 7x^2 + 4x + 5}{(x-2)^2}, \quad x \neq 2.$$

(*a*) The graph of $y = f(x)$ crosses the y-axis at $(0, a)$. State the value of a. **1**

(*b*) For the graph of $f(x)$

(i) write down the equation of the vertical asymptote, **1**

(ii) show algebraically that there is a non-vertical asymptote and state its equation. **3**

(*c*) Find the coordinates and nature of the stationary point of $f(x)$. **4**

(*d*) Show that $f(x) = 0$ has a root in the interval $-2 < x < 0$. **1**

(*e*) Sketch the graph of $y = f(x)$. (You must include on your sketch the results obtained in the first four parts of this question.) **2**

A9. Let $z = \cos\theta + i\sin\theta$.

(*a*) Use the binomial theorem to show that the real part of z^4 is

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Obtain a similar expression for the imaginary part of z^4 in terms of θ. **5**

(*b*) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ. **1**

(*c*) Use your answers to (*a*) and (*b*) to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. **1**

(*d*) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m\theta - \cos^n\theta) + p$ where k, m, n, p are integers. State the values of k, m, n, p. **4**

A10. In a chemical reaction, two substances X and Y combine to form a third substance Z. Let $Q(t)$ denote the number of grams of Z formed t minutes after the reaction begins. The rate at which $Q(t)$ varies is governed by the differential equation

$$\frac{dQ}{dt} = \frac{(30-Q)(15-Q)}{900}.$$

(*a*) Express $\dfrac{900}{(30-Q)(15-Q)}$ in partial fractions. **2**

(*b*) Use your answer to (*a*) to show that the general solution of the differential equation can be written in the form

$$A \ln\left(\frac{30-Q}{15-Q}\right) = t + C,$$

where A and C are constants.

State the value of A and, given that $Q(0) = 0$, find the value of C. **4**

Find, correct to two decimal places,

(i) the time taken to form 5 grams of Z, **1**

(ii) the number of grams of Z formed 45 minutes after the reaction begins. **2**

[*END OF SECTION A*]

Candidates should now attempt ONE of the following

Section B (Mathematics 3) on *Page five*
Section C (Statistics 1) on *Page seven*
Section D (Numerical Analysis 1) on *Page nine*
Section E (Mechanics 1) on *Page eleven*.

Section B (Mathematics 3)

Marks

ONLY candidates doing the course Mathematics 1, 2 and 3 should attempt this Section.

B11. Use the Euclidean Algorithm to find integers of x, y such that

$$195x + 239y = 1.$$ **5**

B12. The $n \times n$ matrix A satisfies the equation

$$A^2 = 5A + 3I$$

where I is the $n \times n$ identity matrix.

Show that A is invertible and express A^{-1} in the form of $pA + qI$. **2**

Obtain a similar expression for A^4. **2**

B13. Use Maclaurin's theorem to write down the expansions, as far as the term in x^3, of

(i) $\sqrt{1+x}$, where $-1 < x < 1$, and **3**

(ii) $(1 - x)^{-2}$, where $-1 < x < 1$. **2**

B14. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$$

in each of the cases

(i) $f(x) = 20\cos x$ **3**

(ii) $f(x) = 20\sin x$ **3**

(iii) $f(x) = 20\cos x + 20\sin x$. **1**

Marks

B15. (*a*) Show that the lines

$$L_1 : \frac{x-3}{2} = \frac{y+1}{3} = \frac{z-6}{1}$$

$$L_2 : \frac{x-3}{-1} = \frac{y-6}{2} = \frac{z-11}{2}$$

intersect, and find the point of intersection. **6**

(*b*) Let A, B, C be the points (2, 1, 0), (3, 3, –1), (5, 0, 2) respectively.

Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

Hence, or otherwise, obtain the equation of the plane containing A, B and C. **5**

[*END OF SECTION B*]

Section C (Statistics 1)

Marks

ONLY candidates doing the course Mathematics 1 and 2 and Statistics 1 should attempt this Section.

C11. An electronics company receives 65% of its components of a particular type from supplier A and the remainder from supplier B. Of the components supplied by A, 2% are defective while 5% of those supplied by B are defective.

Calculate the probability that a randomly selected component, found to be defective, was supplied by A. **5**

C12. Records show a retailer that the number of orders for Noburn toasters during any 28-day shopping period has a Poisson distribution with mean 8.

(*a*) Obtain the probability that, during a randomly selected 28-day shopping period, the number of Noburn toasters ordered exceeds 8. **2**

(*b*) State the number of Noburn toasters which should be in stock at the beginning of a randomly selected 28-day shopping period in order to ensure, with at least 95% probability, that demand during the period is met from that stock. **2**

C13. Find the probability of obtaining at least 55 heads in 100 tosses of a normal coin. **4**

C14. Given a list of all secondary schools in Scotland, outline the steps involved in taking a cluster sample of Higher Mathematics pupils in 5th year. **2**

State two reasons why it might be impractical to take a simple random sample from the target population in this case. **2**

C15. Of 40 fish caught in a loch, 15 were of a certain species.

Find an approximate 95% confidence interval for the proportion of this species in the loch.

Explain what is meant by a 95% confidence interval.

State an assumption that you have made about the 40 fish. **6**

Marks

C16. The management team at a large hospital determined that the time taken, in minutes, to admit a patient was $N(21,4^2)$.

(*a*) Calculate the probability that the mean time to admit a random sample of 10 patients is less than $17{\cdot}7$ minutes. **4**

A new computer system for dealing with patient records was installed at the hospital and the management team was interested in whether any reduction in the mean time to admit patients had resulted. Following the introduction of the system, a random sample of times (minutes) was as follows.

23	20	16	10	20	12	23	18	16	19

(*b*) Conduct an appropriate statistical test to evaluate the evidence for a reduction in the mean waiting time. **5**

[*END OF SECTION C*]

Section D (Numerical Analysis 1)

Marks

ONLY candidates doing the course Mathematics 1 and 2 and Numerical Analysis 1 should attempt this Section.

D11. Obtain the Taylor polynomial of degree two for the function $f(x) = \ln(3x-2),\ x > \frac{2}{3},$ near x. **3**

Estimate the value of $f(1\cdot05)$ using the second degree approximation. **2**

Write down the expression for the principal truncation error in this estimate, and calculate its value. Compare the value of the principal truncation error with the actual error in the approximation. **2**

D12. The following data are available for function f.

x	1	2	3	4
$f(x)$	2·7483	2·3416	1·8409	1·2268

Use the Lagrange interpolation formula to obtain a simplified quadratic expression for $f(x)$, using the values of $f(x)$, where $x = 1$, 2 and 4. **4**

D13. Derive the Newton forward difference formula of degree two. **5**

The following table shows values which have been obtained for a function f.

i	0	1	2	3	4
x	1·6	1·7	1·8	1·9	2·0
$f(x)$	0·826	1·203	1·609	2·042	2·503

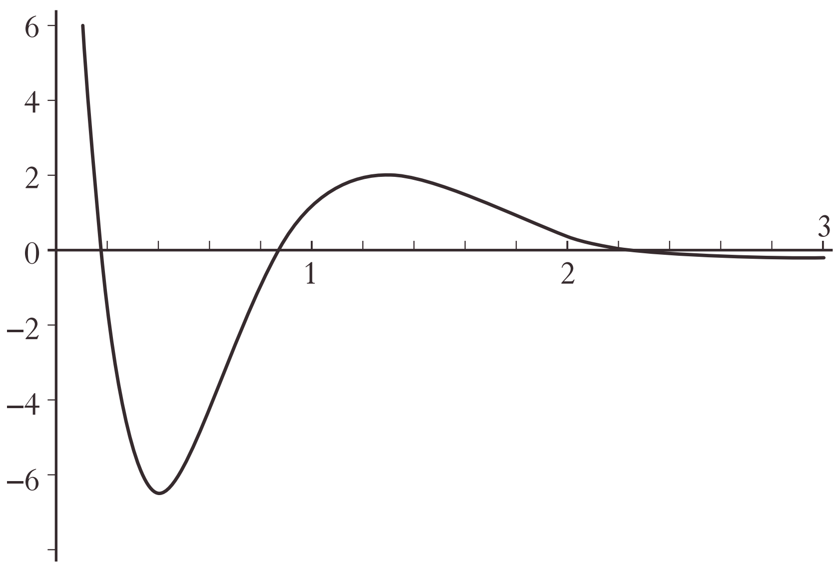
Construct a difference table for f as far as the second differences and use the Newton forward difference formula of degree two to obtain an approximation to $f(1\cdot63)$. **4**

D14. Use Simpson's rule with two strips and the composite Simpson's rule with four strips to obtain two estimates, I_1 and I_2 respectively, for the integral

$$I = \int_1^2 x^4 e^{-2x} dx.$$

Perform the calculations using five decimal places. **4**

The graph of $y = f^{(iv)}(x)$ is shown in the diagram.



The only zero of $f^{(v)}(x)$ on the interval [1, 2] occurs when $x = 1{\cdot}285$ and $f^{(iv)}(1{\cdot}285) = 1{\cdot}903$. Use this information to obtain an estimate of the maximum truncation error in I_2. Hence write down the value of I_2 to a suitable accuracy. **2**

Establish Richardson's formula to improve the accuracy of Simpson's rule by interval halving and use it to obtain an improved estimate I_3 for I based on the values of I_1 and I_2. **6**

[*END OF SECTION D*]

Section E (Mechanics 1)

Marks

ONLY candidates doing the course Mathematics 1 and 2 and Mechanics 1 should attempt this Section.

Where appropriate, candidates should take the magnitude of the acceleration due to gravity as $9{\cdot}8\,\mathrm{m\,s^{-2}}$.

E11. A curler wishes to hit an opponent's stone. Find the minimum speed with which the curler's stone must be projected horizontally towards the opponent's stone for the curler to succeed, given that the coefficient of friction between a curling stone and the level surface of the ice is 0·04 and that the distance from the point of release of the curler's stone to the opponent's stone is 28 metres. **4**

E12. A car travelling at $12\,\mathrm{m\,s^{-1}}$ starts to accelerate 40 metres before leaving a built-up area. The magnitude of the acceleration of the car t seconds later is given by $\frac{1}{3}(13 - 2t)\,\mathrm{m\,s^{-2}}$.

Find the time taken before the speed of the car first reaches $26\,\mathrm{m\,s^{-1}}$ and how far outside the built-up area the car will then be. **5**

E13. A fishing boat sails from a harbour on a bearing of 060° at a constant speed of 26 kilometres per hour. A cargo-ship is sailing on a bearing of 030° at a constant speed of 18 kilometres per hour. The fishing boat leaves the harbour at noon, at which time the cargo ship is 10 kilometres due East of the harbour.

Calculate the time at which the two vessels will be nearest to each other. **7**

E14. A cricketer can throw a ball a **maximum** range of 60 metres to the wicket-keeper. Given that the ball is released from a point 1·5 metres above the ground and caught at the same height, calculate the speed with which it is thrown and the maximum height reached above ground level. (Air resistance may be ignored.) **6**

E15. A packing case of mass 40 kg is to be moved through a vertical distance of 2 metres by means of a ramp of length 6 metres. The coefficient of friction between the packing case and the ramp is such that, if the packing case was placed on the ramp, it would just begin to slide down the ramp. Draw a diagram showing the forces acting on the packing case and calculate this coefficient of friction. **3**

Calculate also the least force which would just move the packing case up the ramp, when the force is applied in the horizontal direction. **7**

[*END OF SECTION E*]

[*END OF QUESTION PAPER*]

[C056/SQP182]

Advanced Higher
Mathematics
Specimen Solutions

NATIONAL
QUALIFICATIONS

Section A (Mathematics 1 and 2)

A1. (*a*) $\dfrac{4}{x^2-4} = \dfrac{4}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$

$= \dfrac{1}{x-2} - \dfrac{1}{x+2}$ **[2]**

(*b*) $\displaystyle\int \frac{x^2}{x^2-4}\,dx = \int 1 + \frac{4}{x^2-4}\,dx$

$\displaystyle= \int 1 + \frac{1}{x-2} - \frac{1}{x+2}\,dx$

$= x + \ln(x-2) - \ln(x+2) + c$ **[4]**

A2. (*a*) $a = 8 + 10t - \frac{3}{4}t^2$

$v = \int 8 + 10t - \frac{3}{4}t^2 dt$

$= 8t + 5t^2 - \frac{1}{4}t^3 + c$

$t = 0;\ v = 0 \Rightarrow c = 0$

$v = 8t + 5t^2 - \frac{1}{4}t^3$ **[2]**

(*b*) $s = \int v\,dt = 4t^2 + \frac{5}{3}t^3 - \frac{1}{16}t^4 + c'$

$t = 0;\ s = 0 \Rightarrow c' = 0$

$\therefore$ when $t = 10,\ s = 400 + \dfrac{5000}{3} - 625 = 1441\dfrac{2}{3}$ **[3]**

A3. $\displaystyle\int_0^2 \frac{x+1}{\sqrt{16-x^2}}dx$ $\qquad x = 4\sin t$

$$= \int_0^{\pi/6} \frac{4\sin t + 1}{\sqrt{16-16\sin^2 t}} 4\cos t\, dt \qquad \Rightarrow \frac{dx}{dt} = 4\cos t$$

$x = 0 \Rightarrow t = 0;$

$$= \int_0^{\pi/6} \frac{(4\sin t + 1) \times 4\cos t}{4\cos t}\, dt \qquad x = 2 \Rightarrow t = \frac{\pi}{6}$$

$$= \int_0^{\pi/6} (4\sin t + 1)dt$$

$$= [-4\cos t + t]_0^{\pi/6} = 2\sqrt{3} + 4 + \frac{\pi}{6} \approx 1\cdot059$$ **[5]**

A4.

$$\left.\begin{array}{ccc} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{array}\right| \begin{array}{r} 0 \\ -1\cdot1 \\ 0\cdot9 \end{array}$$

$$\left.\begin{array}{ccc} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 2 & 1 \end{array}\right| \begin{array}{r} 0 \\ -1\cdot1 \\ 0\cdot9 \end{array} \qquad \begin{array}{l} \\ (r_2' = r_2 - 2r_1) \\ (r_3' = r_3 - r_1) \end{array}$$

$$\left.\begin{array}{ccc} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{array}\right| \begin{array}{r} 0 \\ -1\cdot1 \\ 0\cdot5 \end{array} \qquad \begin{array}{l} \\ \\ (r_3'' = 3r_3 + 2r_2) \end{array}$$

Hence $z = 0\cdot5$; $y = (1\cdot1 - 0\cdot5)/3 = 0\cdot2$;
$x = -0\cdot2 - 0\cdot5 = -0\cdot7$ **[5]**

Marks

A5. (*a*) $x^2 + xy + y^2 = 1$

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$$ **[2]**

(*b*) (i) $x = 2t + 1; \qquad y = 2t(t-1)$

$$\frac{dx}{dt} = 2; \frac{dy}{dt} = 4t - 2 \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2t - 1$$ **[2]**

(ii) $t = \frac{1}{2}(x-1) \qquad y = (x-1)\left[\frac{1}{2}(x-1)-1\right]$

$$= \frac{1}{2}(x-1)(x-3)$$ **[1]**

A6. (*a*)

$$\begin{aligned} u_3 &= 2d + u_1 = 5 \\ 2d &= 5 - 45 \\ d &= -20 \\ u_{11} &= 45 + 10(-20) \\ &= -155 \end{aligned}$$ **[2]**

(*b*) $45r^2 = 5$

$r = \frac{1}{3}$ since v_1, ... are positive

$$S = \frac{45}{1-\frac{1}{3}} = 67\tfrac{1}{2}$$ **[3]**

A7. $n = 1$ LHS $= 1 \times 2 = 2$

RHS $= \frac{1}{3} \times 1 \times 2 \times 3 = 2$

True for $n = 1$.

Assume true for n and consider

$$\begin{aligned} \sum_{r=1}^{n+1} r(r+1) &= \sum_{r=1}^{n} r(r+1) + (n+1)(n+2) \\ &= \tfrac{1}{3}n(n+1)(n+2) + (n+1)(n+2) \\ &= \tfrac{1}{3}n(n+1)(n+2)(n+3) \end{aligned}$$

Thus if true for n then true for $n + 1$.

Therefore since true for $n = 1$, true for all $n \geq 1$. **[5]**

A8. $f(x) = \dfrac{2x^3 - 7x^2 + 4x + 5}{(x-2)^2}$

(a) $x = 0 \Rightarrow y = \frac{5}{4} \Rightarrow a = \frac{5}{4}$ **[1]**

(b) (i) $x = 2$ **[1]**

(ii) After division, the function can be expressed in quotient/remainder form:

$$f(x) = 2x + 1 + \frac{1}{(x-2)^2}$$

Thus the line $y = 2x + 1$ is a slant asymptote. **[3]**

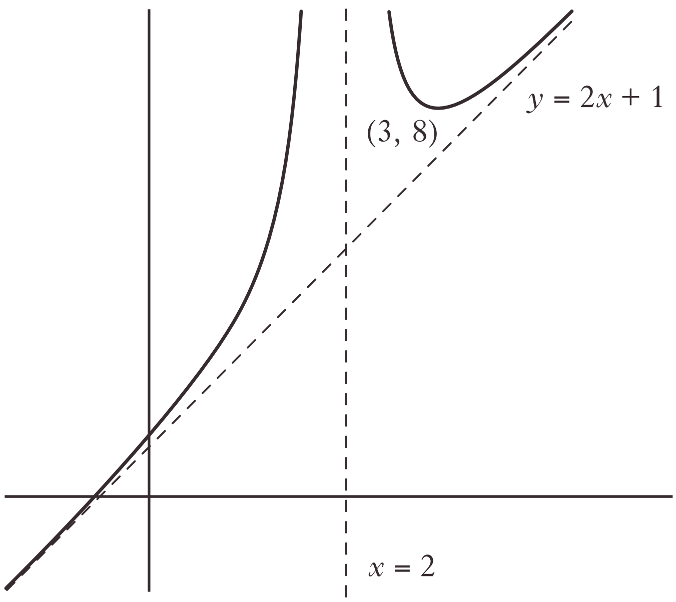
(c) From (b), $f'(x) = 2 - \dfrac{2}{(x-2)^3}$. Turning point when

$$2 - \frac{2}{(x-2)^3} = 0$$
$$(x-2)^3 = 1$$
$$x - 2 = 1 \Rightarrow x = 3$$
$$f''(x) = \frac{6}{(x-2)^4} > 0 \text{ for all } x.$$

The stationary point at (3, 8) is a minimum turning point. **[4]**

(d) $f(-2) = \dfrac{-16-28-8+5}{(-4)^2} < 0$; $f(0) = \dfrac{5}{4} > 0$.
Hence a root between -2 and 0. **[1]**

(e)



[2]

Marks

A9. (*a*) $z^4 = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i\,(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

Hence the real part is $\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

The imaginary part is $(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

$= 4\cos\theta\sin\theta\,(\cos^2\theta - \sin^2\theta)$ **[5]**

(*b*) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ **[1]**

(*c*) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$ **[1]**

(*d*) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta\,(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

$= 8\,(\cos^4\theta - \cos^2\theta) + 1$

ie $k = 8$, $m = 4$, $n = 2$, $p = 1$. **[4]**

Marks

A10. (*a*) $900 = A(15 - Q) + B(30 - Q)$

Letting $Q = 30$ gives $A = -60$

and $Q = 15$ gives $B = 60$

$$\frac{900}{(30-Q)(15-Q)} = \frac{-60}{(30-Q)} + \frac{60}{(15-Q)}$$ **[2]**

(*b*) $\frac{dQ}{dt} = \frac{(30-Q)(15-Q)}{900}$

$$\therefore \int \frac{900}{(30-Q)(15-Q)} dQ = \int dt$$

$$\therefore \int \frac{-60}{(30-Q)} + \frac{60}{(15-Q)} dQ = \int dt$$

$$60 \ln(30-Q) - 60 \ln(15-Q) = t + C$$

ie $60 \ln\left(\frac{30-Q}{15-Q}\right) = t + C$

$A = 60$

$C = 60 \ln 2 = 41 \cdot 59$ to 2 decimal places **[4]**

(i) $t = 60 \ln\left(\frac{30-Q}{15-Q}\right) - 60 \ln 2 = 60 \ln\left(\frac{30-Q}{2(15-Q)}\right)$

When $Q = 5$, $t = 60 \ln \frac{25}{20} = 13 \cdot 39$ minutes to 2 decimal places. **[1]**

(ii) $\ln\left(\frac{30-Q}{2(15-Q)}\right) = \frac{t}{60}$

$$30 - Q = 2(15-Q)e^{t/60}$$

$$Q(2e^{t/60} - 1) = 30(e^{t/60} - 1)$$

$$Q = \frac{30(e^{t/60} - 1)}{2e^{t/60} - 1}$$

When $t = 45, Q = 10 \cdot 36$ grams to 2 decimal places. **[2]**

Section B (Mathematics 3)

B11.

$$\begin{aligned}
239 &= 1 \times 195 + 44 \\
195 &= 4 \times 44 + 19 \\
44 &= 2 \times 19 + 6 \\
19 &= 3 \times 6 + 1 \\
\text{So } 1 &= 19 - 3 \times 6 \\
&= 19 - 3(44 - 2 \times 19) \\
&= 7 \times (195 - 4 \times 44) - 3 \times 44 \\
&= 7 \times 195 - 31(239 - 195) \\
&= 38 \times 195 - 31 \times 239
\end{aligned}$$

ie $195x + 239y = 1$ when $x = 38$ and $y = -31$ **[5]**

B12.

$$\begin{aligned}
A^2 &= 5A + 3I \\
\therefore A^2 - 5A &= 3I \\
A(\tfrac{1}{3}A - \tfrac{5}{3}I) &= I
\end{aligned}
\qquad
\begin{aligned}
A^4 &= (5A + 3I)^2 \\
&= 25A^2 + 30A + 9I \\
&= 155A + 84I
\end{aligned}$$

$\therefore A$ is invertible and $A^{-1} = \frac{1}{3}(A - 5I)$ **[2, 2]**

B13. (i)

$$\begin{aligned}
f(x) &= \sqrt{1+x} & f(0) &= 1 \\
&= (1+x)^{1/2} \\
f'(x) &= \frac{1}{2}(1+x)^{-1/2} & f'(0) &= \frac{1}{2} \\
f''(x) &= -\frac{1}{4}(1+x)^{-3/2} & f''(0) &= -\frac{1}{4} \\
f'''(x) &= \frac{3}{8}(1+x)^{-5/2} & f'''(0) &= \frac{3}{8}
\end{aligned}$$

$\therefore \sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$ **[3]**

(ii)

$$\begin{aligned}
f(x) &= (1-x)^{-2} & f(0) &= 1 \\
f'(x) &= 2(1-x)^{-3} & f'(0) &= 2 \\
f''(x) &= 6(1-x)^{-4} & f''(0) &= 6 \\
f'''(x) &= 24(1-x)^{-5} & f'''(0) &= 24
\end{aligned}$$

$\therefore (1-x)^{-2} \approx 1 + 2x + 3x^2 + 4x^3$ **[2]**

B14. $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$$

A.E. $m^2 - 5m + 6 = 0$

$\therefore m = 2$ or $m = 3$

C.F. $y = Ae^{2x} + Be^{3x}$

(i) $f(x) = 20\cos x$; P.I. $= a\cos x + b\sin x$

$\Rightarrow -a\cos x - b\sin x + 5a\sin x - 5b\cos x + 6a\cos x + 6b\sin x = 20\cos x$

$5a - 5b = 20$

$5a + 5b = 0 \Rightarrow a = -b$

$-10b = 20 \Rightarrow b = -2;\ a = 2$

Solution $y = Ae^{2x} + Be^{3x} + 2\cos x - 2\sin x$ **[3]**

(ii) $f(x) = 20\sin x$; P.I. $= c\cos x + d\sin x$

$5c - 5d = 0 \Rightarrow c = d$

$5c + 5d = 20 \Rightarrow c = d = 2$

Solution $y = Ae^{2x} + Be^{3x} + 2\cos x + 2\sin x$ **[3]**

(iii) $f(x) = 20\cos x + 20\sin x$

Solution $y = Ae^{2x} + Be^{3x} + 4\cos x$ **[1]**

B15. (*a*) L_1: $x = 3 + 2s$; $y = -1 + 3s$; $z = 6 + s$

L_2: $x = 3 - t$; $y = 6 + 2t$; $z = 11 + 2t$

$\therefore$ for x: $3 + 2s = 3 - t \Rightarrow t = -2s$

$\therefore$ for y: $3s - 1 = 6 + 2t$

$7\lambda = 7 \Rightarrow s = 1$; $t = -2$

$\therefore L_1$: $x = 5$; $y = 2$; $z = 6 + s = 7$

$\therefore L_2$: $x = 5$; $y = 2$; $z = 11 + 2t = 11 - 4 = 7$

ie L_1 and L_2 intersect at (5, 2, 7) **[6]**

(*b*) $A(2,1,0)$; $B(3,3,-1)$; $C(5,0,2)$

$\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$; $\overrightarrow{AC} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

Equation of plane has form $3x - 5y - 7z = k$

$(2, 1, 0) \Rightarrow k = 1$

Equation is $3x - 5y - 7z = 1$. **[5]**

Section C (Statistics 1)

C11. $P(A) = 0\cdot65 \quad P(def|A) = 0\cdot02 \quad$ Bayes Th. $P(A|def) = \dfrac{0\cdot02\times0\cdot65}{0\cdot02\times0\cdot65+0\cdot05\times0\cdot35} = \dfrac{26}{61}$

$P(B) = 0\cdot35 \quad P(def|B) = 0\cdot05$ **[5]**

C12. $P(X>8) = 1 - p(X\le 8) = 0\cdot407$ **[2]**

$\left.\begin{array}{l} P(X\le 12) = 0\cdot936 \\ P(X\le 13) = 0\cdot966 \end{array}\right\}$ 13 required **[2]**

C13. $B\left(100,\frac{1}{2}\right) \quad P(B\ge 55) \approx P\left(Z \ge \dfrac{55-\frac{1}{2}-50}{5}\right) = 0\cdot184$ **[4]**

C14. Take a random sample of all schools with Higher Mathematics candidates and then select every 5th year pupil, taking Higher Mathematics, from those schools. **[2]**

A list of all Higher Mathematics candidates would not be available to us and there would be a high cost in terms of time and money if we had to visit widely scattered schools. **[2]**

C15. $\hat{p} = 0\cdot375$

$\hat{q} = 0\cdot625 \qquad$ 95% C.I. is $0\cdot375 \pm 1\cdot96\sqrt{\dfrac{0\cdot375\times0\cdot625}{40}}$

$n = 40 \qquad\qquad = 0\cdot225 \rightarrow 0\cdot525$

For every one hundred intervals calculated we would expect 95 of them to capture the true value of p and 5 not to.

We must assume that the 40 fish constitute a random sample. **[6]**

C16. $\overline{X} \sim N\left(21,\left(\dfrac{4}{\sqrt{10}}\right)^2\right) \qquad P(\overline{X} < 17\cdot7) = P\left(Z < \dfrac{17\cdot7-21}{\frac{4}{\sqrt{10}}}\right) = 0\cdot005$ **[4]**

$\overline{X} = 17\cdot7 \qquad H_0 : \mu = 21 \qquad H_1 : \mu < 21 \qquad$ 1 – tail test

$P(\overline{X} \le 17\cdot7) = 0\cdot005 < 0\cdot01$

Reject H_0 at 1% level ie there is strong evidence of a reduction in waiting time. **[5]**

Section D (Numerical Analysis 1)

D11. $f(x) = \ln(3x-2); \quad f'(x) = \dfrac{3}{3x-2}; \quad f''(x) = \dfrac{-9}{(3x-2)^2}$

$$f'''(x) = \frac{54}{(3x-2)^3}$$

2nd degree polynomial is

$$p_2(x) = p_2(1+h) = f(1) + hf'(1) + \frac{h^2}{2}f''(1)$$

$$= \ln 1 + 3h - 4\cdot 5h^2 = 3h - 4\cdot 5h^2$$ **[3]**

At $x = 1\cdot 05,\ h = 0\cdot 05$ so $f(1\cdot 05) \approx 0\cdot 1388$ **[2]**

Principal error $= \dfrac{h^3}{3!}\dfrac{54}{(3-2)^3} = 0\cdot 0011$

Actual error $= 0\cdot 1388 - \ln 1\cdot 15 = 0\cdot 00096$

Values are equal to 3 decimal places **[2]**

D12. $L(x) = \dfrac{(x-2)(x-4)}{(-1)(-3)}\,2\cdot 7483 + \dfrac{(x-1)(x-4)}{1(-2)}\,2\cdot 3416 + \dfrac{(x-1)(x-2)}{3.2}\,1\cdot 2268$

$$= (x^2-6x+8)\,\frac{2\cdot 7483}{3} - (x^2-5x+4)\,\frac{2\cdot 3416}{2} + (x^2-3x+2)\,\frac{1\cdot 2268}{6}$$

$$= -0\cdot 0502x^2 - 0\cdot 2560 + 3\cdot 0545$$ **[4]**

D13. Consider the quadratic through $(x_0, f_0), (x_1, f_1), (x_2, f_2)$

Let equation be $y = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1)$

Then $f_0 = A_0; \quad f_1 = A_0 + A_1 h; \quad A_2 = \dfrac{f_2 - 2f_1 + f_0}{2h^2} = \dfrac{\Delta^2 f_0}{2h^2}$

Thus $y = f_0 + \dfrac{x-x_0}{h}\Delta f_0 + \dfrac{(x-x_0)(x-x_1)}{2h^2}\Delta^2 f_0$

Setting $x = x_0 + ph$, when $0 < p < 1$, gives

$$y = f_0 + p\Delta f_0 + \frac{p(p-1)}{2}\Delta^2 f_0$$ **[5]**

x_i	f_i	Δf_i	$\Delta^2 f_i$
1·6	0·826		
		377	
1·7	1·203		29
		406	
1·8	1·609		27
		433	
1·9	2·042		28
		461	
2·0	2·503		

$$p = \frac{0\cdot 03}{0\cdot 1} = 0\cdot 3$$

$f(1\cdot 63) \approx 0\cdot 826 + 0\cdot 3 \times 0\cdot 377 + \dfrac{0\cdot 3.(-0\cdot 7)}{2}0\cdot 029 = 0\cdot 936$ **[4]**

Marks

D14.

x	$f(x)$	m	$mf(x)_4$	m	$mf(x)_2$
1·00	0·13534	1	0·13534	1	0·13534
1·25	0·20040	4	0·80160		
1·50	0·25205	2	0·50410	4	1·00820
1·75	0·28322	4	1·13288		
2·00	0·29305	1	0·29305	1	0·29305
			2·86697		1·43659

$I_1 = \frac{1 \cdot 43659}{6} = 0 \cdot 23943$ $\qquad$ $I_2 = \frac{2 \cdot 86697}{12} = 0 \cdot 23891$ **[4]**

$|E| \leq \frac{1}{180} \times 0 \cdot 25^4 \times 1 \cdot 903 = 4 \cdot 13 \times 10^{-5} = 0 \cdot 000041$

Hence $I_2 = 0 \cdot 2389$. **[2]**

With n strips and width $2h$, the Taylor series for an integral approximated by Simpson's rule, with principal truncation error $O(h^4)$, is

$I = I_n + C(2h)^4 + D(2h)^6 + \ldots. \quad = I_n + 16Ch^4 + \ldots. \qquad -(1)$

Similarly, with $2n$ strips of width h,

$I = I_{2n} + Ch^4 + Dh^6 + \ldots \quad -(2)$

Taking $16 \times (2) - (1)$ gives $15I = 16I_{2n} - I_n + O(h^6)$

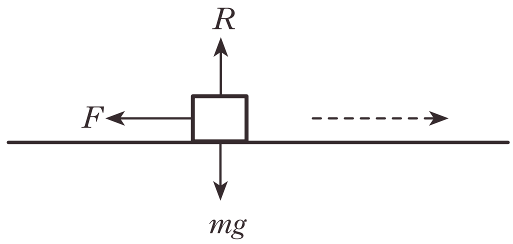
$$\Rightarrow I = I_{2n} + \frac{1}{15}(I_{2n} - I_n)$$ **[4]**

and $I_3 = 0 \cdot 23891 + \frac{1}{15}(-0 \cdot 00052) = 0 \cdot 23888$ **[2]**

Marks

Section E (Mechanics 1)

E11.



$R = mg$

$ma = -F = -\mu R$

$ma = -0 \cdot 04mg$

$a = -0 \cdot 04g$

$v^2 = u^2 + 2as$

$0 = u^2 - 2 \times 0 \cdot 04g \times 28$

$u^2 = 21 \cdot 952 \Rightarrow u = 4 \cdot 7 \text{ m s}^{-1}$ [to 1 decimal place] **[4]**

E12. $a = \frac{1}{3}(13 - 2t)$

$v = \frac{1}{3}(13t - t^2) + c$

$v = 12 \quad t = 0 \Rightarrow c = 12 \Rightarrow v = \frac{1}{3}(13t - t^2) + 12$

$\frac{1}{3}(13t - t^2) + 12 = 26 \Rightarrow t^2 - 13t + 42 = 0$

$(t - 6)(t - 7) = 0 \quad t = 6, 7$

First reaches 26 m s^{-1} after 6 secs.

$s = \frac{1}{3}\left(\frac{13t^2}{2} - \frac{t^3}{3}\right) + 12t + c$

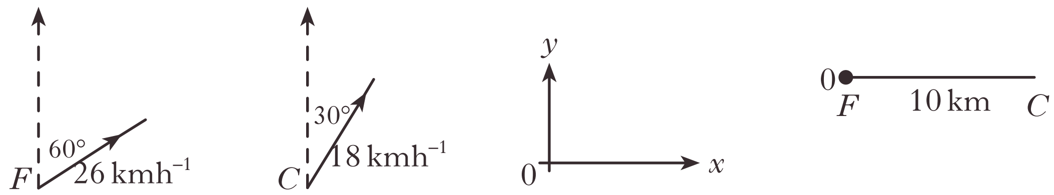
$s = -40 \quad t = 0 \Rightarrow c = -40$

$s = \frac{13t^2}{6} - \frac{t^3}{9} + 12t - 40$

$t = 6 \quad s = 78 - 24 + 72 - 40 = 86 \text{ m}$ outside the built-up area. **[5]**

Marks

E13.



$\mathbf{v}_C = 18\sin 30\,\mathbf{i} + 18\cos 30\,\mathbf{j} = 9\mathbf{i} + 15\cdot 6\,\mathbf{j}$

$\mathbf{r}_F = 22\cdot 5t\mathbf{i} + 13t\mathbf{j}$ $\qquad\qquad$ $\mathbf{r}_c = (9t + 10)\mathbf{i} + 15\cdot 6t\,\mathbf{j}$

$\Rightarrow \mathbf{r}_F - \mathbf{r}_C = (13\cdot 5t - 10)\mathbf{i} - 2\cdot 6t\mathbf{j}$

$\left|\mathbf{r}_F - \mathbf{r}_C\right|^2 = (13\cdot 5t - 10)^2 + 6\cdot 76t^2$

$\dfrac{d\left|\mathbf{r}_F - \mathbf{r}_C\right|^2}{dt} = 27(13\cdot 5t - 10) + 13\cdot 5t = 0$ for stationary point

$\Rightarrow 378t = 270$

$t = 0\cdot 71\,\text{h}$

$= 43\text{min}$

Vessels will be closest at 12.43 pm **[7]**

E14. $R = \dfrac{u^2\sin 2\alpha}{g} \Rightarrow$ Max range $= \dfrac{u^2}{g} = 60 \Rightarrow u^2 = 60\text{g}$

$u = 24\cdot 2\text{ m s}^{-1}$

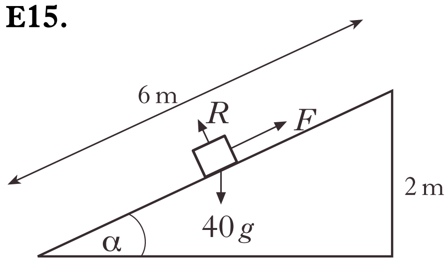
Max height when $0 = u^2\sin^2\alpha - 2gh$

$0 = 60\text{ g} \times \dfrac{1}{2} - 2gh$

$h = \dfrac{30g}{2g} = 15\,\text{m}$

So max height above ground $= 16\cdot 5$ m **[6]**

E15.



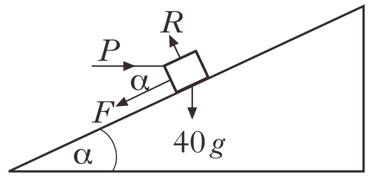
Perp to plane	R	=	$40\,g\cos\alpha$
Parallel to plane	F	=	$40\,g\sin\alpha$

On point of slipping $\quad F = \mu R$

$$\Rightarrow \mu 40\,g\cos\alpha = 40\,g\sin\alpha$$

$$\mu = \tan\alpha = \frac{2}{\sqrt{32}} = \frac{2}{4\sqrt{2}} = \frac{\sqrt{2}}{4} \text{ (or } 3\cdot5)$$

[3]



Perp to plane	R	=	$40\,g\cos\alpha + P\sin\alpha$
Parallel to plane	$F + 40\,g\sin\alpha$	=	$P\cos\alpha$

$$F = \frac{\sqrt{2}}{4}R \Rightarrow \frac{\sqrt{2}}{4}\left(40g\frac{\sqrt{32}}{6} + P\frac{1}{3}\right) + 40\,g\,.\,\frac{1}{3} = P\,\frac{\sqrt{32}}{6}$$

$$\frac{40g}{3} + \frac{P\sqrt{2}}{12} + \frac{40\,g}{3} = \frac{2\sqrt{2}P}{3}$$

(candidates likely to use μ = 0·35 and numerical value of α)

$$80\,g = (2 - \tfrac{1}{4})\sqrt{2}P$$

$$P = \frac{80 \times 9\cdot8}{1\cdot75 \times \sqrt{2}} = 316\cdot8\text{ N}$$

[7]

[*END OF MARKING INSTRUCTIONS*]