[C100/SQP255]

Mathematics
Advanced Higher

Time: 3 hours

NATIONAL QUALIFICATIONS

Specimen Question Paper
for use in and after 2004

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions .
3. **Full credit will be given only where the solution contains appropriate working.**

Answer all the questions.

Marks

1. (*a*) Find partial fractions for

$$\frac{4}{x^2 - 4}.$$ **2**

(*b*) By using (*a*) obtain

$$\int \frac{x^2}{x^2 - 4}\,dx.$$ **4**

2. Use the Euclidean Algorithm to find integers of x, y such that

$$195x + 239y = 1.$$ **5**

3. The performance of a prototype surface-to-air missile was measured on a horizontal test bed at the firing range and it was found that, until its fuel was exhausted, its acceleration (measured in $\mathrm{m\,s^{-2}}$) t seconds after firing was given by

$$a = 8 + 10t - \frac{3}{4}t^2.$$

(*a*) Obtain a formula for its speed, t seconds after firing. **2**

(*b*) The missile contained enough fuel for 10 seconds. What horizontal distance would it have covered on the firing range when its fuel was exhausted? **3**

4. The $n \times n$ matrix A satisfies the equation

$$A^2 = 5A + 3I$$

where I is the $n \times n$ identity matrix.

Show that A is invertible and express A^{-1} in the form of $pA + qI$. **2**

Obtain a similar expression for A^4. **2**

5. Use the substitution $x = 4\sin t$ to evaluate the definite integral

$$\int_0^2 \frac{x+1}{\sqrt{16 - x^2}}\,dx.$$ **5**

Marks

6. Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= -1 \cdot 1 \\ x + 3y + 2z &= 0 \cdot 9. \end{aligned}$$ **5**

7. Use Maclaurin's theorem to write down the expansions, as far as the term in x^3, of

(i) $\sqrt{1+x}$, where $-1 < x < 1$, and **3**

(ii) $(1-x)^{-2}$, where $-1 < x < 1$. **2**

8. (*a*) Find the derivative of y with respect to x, where y is defined as an implicit function of x by the equation

$$x^2 + xy + y^2 = 1.$$ **2**

(*b*) A curve is defined by the parametric equations

$$x = 2t + 1, \qquad y = 2t(t-1).$$

(i) Find $\dfrac{dy}{dx}$ in terms of t. **2**

(ii) Eliminate t to find y in terms of x. **1**

9. Let $u_1, u_2, \ldots, u_n, \ldots$ be an arithmetic sequence and $v_1, v_2, \ldots, v_n, \ldots$ be a geometric sequence. The first terms u_1 and v_1 are both equal to 45, and the third terms u_3 and v_3 are both equal to 5.

(*a*) Find u_{11}. **2**

(*b*) Given that $v_1, v_2, \ldots$ is a sequence of **positive** numbers, calculate $\sum_{n=1}^{\infty} v_n$. **3**

10. Use induction to prove that

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

for all positive integers n. **5**

Marks

11. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$$

in each of the cases

(i) $f(x) = 20\cos x$ **3**

(ii) $f(x) = 20\sin x$ **3**

(iii) $f(x) = 20\cos x + 20\sin x$. **1**

12. Let the function f be given by

$$f(x) = \frac{2x^3 - 7x^2 + 4x + 5}{(x-2)^2}, \quad x \neq 2.$$

(*a*) The graph of $y = f(x)$ crosses the y-axis at $(0, a)$. State the value of a. **1**

(*b*) For the graph of $f(x)$

(i) write down the equation of the vertical asymptote, **1**

(ii) show algebraically that there is a non-vertical asymptote and state its equation. **3**

(*c*) Find the coordinates and nature of the stationary point of $f(x)$. **4**

(*d*) Show that $f(x) = 0$ has a root in the interval $-2 < x < 0$. **1**

(*e*) Sketch the graph of $y = f(x)$. (You must include on your sketch the results obtained in the first four parts of this question.) **2**

13. (*a*) Show that the lines

$$L_1 : \frac{x-3}{2} = \frac{y+1}{3} = \frac{z-6}{1}$$

$$L_2 : \frac{x-3}{-1} = \frac{y-6}{2} = \frac{z-11}{2}$$

intersect, and find the point of intersection. **6**

(*b*) Let A, B, C be the points $(2, 1, 0)$, $(3, 3, -1)$, $(5, 0, 2)$ respectively.

Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

Hence, or otherwise, obtain the equation of the plane containing A, B and C. **5**

14. Let $z = \cos\theta + i\sin\theta$.

Marks

(a) Use the binomial theorem to show that the real part of z^4 is

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Obtain a similar expression for the imaginary part of z^4 in terms of θ. **5**

(b) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ. **1**

(c) Use your answers to (a) and (b) to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. **1**

(d) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m\theta - \cos^n\theta) + p$ where k, m, n, p are integers. State the values of k, m, n, p. **4**

15. In a chemical reaction, two substances X and Y combine to form a third substance Z. Let $Q(t)$ denote the number of grams of Z formed t minutes after the reaction begins. The rate at which $Q(t)$ varies is governed by the differential equation

$$\frac{dQ}{dt} = \frac{(30-Q)(15-Q)}{900}.$$

(a) Express $\dfrac{900}{(30-Q)(15-Q)}$ in partial fractions. **2**

(b) Use your answer to (a) to show that the general solution of the differential equation can be written in the form

$$A\ln\left(\frac{30-Q}{15-Q}\right) = t + C,$$

where A and C are constants.

State the value of A and, given that $Q(0) = 0$, find the value of C. **4**

Find, correct to two decimal places,

(i) the time taken to form 5 grams of Z, **1**

(ii) the number of grams of Z formed 45 minutes after the reaction begins. **2**

[*END OF SPECIMEN QUESTION PAPER*]

[C100/SQP255]

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Marks

1. (*a*) $\frac{4}{x^2-4} = \frac{4}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x-2}$

$= \frac{1}{x-2} - \frac{1}{x+2}$ **[2]**

(*b*) $\int \frac{x^2}{x^2-4}dx = \int 1 + \frac{4}{x^2-4}dx$

$= \int 1 + \frac{1}{x-2} - \frac{1}{x+2}dx$

$= x + \ln(x-2) - \ln(x+2) + c$ **[4]**

2.

$239 = 1 \times 195 + 44$

$195 = 4 \times 44 + 19$

$44 = 2 \times 19 + 6$

$19 = 3 \times 6 + 1$

So $1 = 19 - 3 \times 6$

$= 19 - 3(44 - 2 \times 19)$

$= 7 \times (195 - 4 \times 44) - 3 \times 44$

$= 7 \times 195 - 31(239 - 195)$

$= 38 \times 195 - 31 \times 239$

ie $195x + 239y = 1$ when $x = 38$ and $y = -31$ **[5]**

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3. (*a*)

$a = 8 + 10t - \frac{3}{4}t^2$

$v = \int 8 + 10t - \frac{3}{4}t^2 dt$

$= 8t + 5t^2 - \frac{1}{4}t^3 + c$

$t = 0; v = 0 \Rightarrow c = 0$

$v = 8t + 5t^2 - \frac{1}{4}t^3$ **[2]**

(*b*)

$s = \int v\, dt = 4t^2 + \frac{5}{3}t^3 - \frac{1}{16}t^4 + c'$

$t = 0; s = 0 \Rightarrow c' = 0$

$\therefore$ when $t = 10$, $s = 400 + \frac{5000}{3} - 625 = 1441\frac{2}{3}$ **[3]**

Marks

4. $A^2 = 5A + 3I$

$\therefore A^2 - 5A = 3I$

$A(\frac{1}{3}A - \frac{5}{3}I) = I$

$\therefore A$ is invertible and $A^{-1} = \frac{1}{3}(A - 5I)$

$A^4 = (5A+3I)^2$

$= 25A^2 + 30A + 9I$

$= 155A + 84I$

[2, 2]

5. $\displaystyle\int_0^2 \frac{x+1}{\sqrt{16-x^2}}\,dx$

$x = 4\sin t$

$\Rightarrow \frac{dx}{dt} = 4\cos t$

$x = 0 \Rightarrow t = 0;$

$x = 2 \Rightarrow t = \frac{\pi}{6}$

$$= \int_0^{\pi/6} \frac{4\sin t + 1}{16 - 16\sin^2}\,4\cos t\;dt$$

$$= \int_0^{\pi/6} \frac{(4\sin t + 1)\times 4\cos t}{4\cos t}\,dt$$

$$= \int_0^{\pi/6} (4\sin t + 1)\,dt$$

$$= [-4\cos t + t]_0^{\pi/6} = 2\sqrt{3} + 4 + \frac{\pi}{6} \approx 1\cdot059$$

[5]

6.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -1\cdot1 \\ 1 & 3 & 2 & 0\cdot9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1\cdot1 \\ 0 & 2 & 1 & 0\cdot9 \end{array}\right) \quad \begin{array}{l}(r_2' = r_2 - 2r_1) \\ (r_3' = r_3 - r_1)\end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1\cdot1 \\ 0 & 0 & 1 & 0\cdot5 \end{array}\right) \quad (r_3'' = 3r_3 + 2r_2)$$

Hence $z = 0.5$; $y = (1\cdot1 - 0\cdot5)/3 = 0\cdot2$;
$x = -0\cdot2 - 0\cdot5 = -0\cdot7$

[5]

Marks

7. (i)

$$f(x) = \sqrt{1+x} \qquad f(0) = 1$$
$$= (1+x)^{1/2}$$
$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f'(0) = \frac{1}{2}$$
$$f''(x) = -\frac{1}{4}(1+x)^{-3/2} \qquad f''(0) = -\frac{1}{4}$$
$$f'''(x) = \frac{3}{8}(1+x)^{-5/2} \qquad f'''(0) = \frac{3}{8}$$
$$\therefore \sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

[3]

(ii)

$$f(x) = (1-x)^{-2} \qquad f(0) = 1$$
$$f'(x) = 2(1-x)^{-3} \qquad f'(0) = 2$$
$$f''(x) = 6(1-x)^{-4} \qquad f''(0) = 6$$
$$f'''(x) = 24(1-x)^{-5} \qquad f'''(0) = 24$$
$$\therefore (1-x)^{-2} \approx 1 + 2x + 3x^2 + 4x^3$$

[2]

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8. (*a*)

$$x^2 + xy + y^2 = 1$$
$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$$

[2]

(*b*) (i)

$$x = 2t + 1; \qquad y = 2t(t-1)$$
$$\frac{dx}{dt} = 2;\ \frac{dy}{dt} = 4t - 2 \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2t - 1$$

[2]

(ii)

$$t = \frac{1}{2}(x-1) \qquad y = (x-1)\left[\frac{1}{2}(x-1) - 1\right]$$
$$= \frac{1}{2}(x-1)(x-3)$$

[1]

Marks

9. (*a*)

$$u_3 = 2d + u_1 = 5$$
$$2d = 5 - 45$$
$$d = -20$$
$$u_{11} = 45 + 10(-20)$$
$$= -155$$

[2]

(*b*)

$$45r^2 = 5$$

$r = \frac{1}{3}$ since v_1, ... are positive

$$S = \frac{45}{1-\frac{1}{3}} = 67\tfrac{1}{2}$$

[3]

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10. $n = 1$ LHS $= 1 \times 2 = 2$

RHS $= \frac{1}{3} \times 1 \times 2 \times 3 = 2$

True for $n = 1$.

Assume true for k and consider

$$\sum_{r=1}^{k+1} r(r+1) = \sum_{r=1}^{k} r(r+1) + (k+1)(k+2)$$
$$= \tfrac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$
$$= \tfrac{1}{3}(k+1)(k+2)(k+3)$$

Thus if true for k then true for $k+1$.

Therefore since true for $n = 1$, true for all $n \geq 1$.

[5]

Marks

11. $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$$

A.E. $m^2 - 5m + 6 = 0$

$\therefore\ m = 2$ or $m = 3$

C.F. $y = Ae^{2x} + Be^{3x}$

(i) $f(x) = 20\cos x$; P.I. $= a\cos x + b\sin x$

$\Rightarrow -a\cos x - b\sin x + 5a\sin x - 5b\cos x + 6a\cos x + 6b\sin x = 20\cos x$

$5a - 5b = 20$

$5a + 5b = 0 \Rightarrow a = -b$

$-10b = 20 \Rightarrow b = -2;\ a = 2$

Solution $y = Ae^{2x} + Be^{3x} + 2\cos x - 2\sin x$ **[3]**

(ii) $f(x) = 20\sin x$; P.I. $= c\cos x + d\sin x$

$5c - 5d = 0 \Rightarrow c = d$

$5c + 5d = 20 \Rightarrow c = d = 2$

Solution $y = Ae^{2x} + Be^{3x} + 2\cos x + 2\sin x$ **[3]**

(iii) $f(x) = 20\cos x + 20\sin x$

Solution $y = Ae^{2x} + Be^{3x} + 4\cos x$ **[1]**

Marks

12. $f(x) = \dfrac{2x^3 - 7x^2 + 4x + 5}{(x-2)^2}$

(a) $x = 0 \Rightarrow y = \frac{5}{4} \Rightarrow a = \frac{5}{4}$ **[1]**

(b) (i) $x = 2$ **[1]**

(ii) After division, the function can be expressed in quotient/remainder form:

$$f(x) = 2x + 1 + \frac{1}{(x-2)^2}$$

Thus the line $y = 2x + 1$ is a slant asymptote. **[3]**

(c) From (b), $f'(x) = 2 - \dfrac{2}{(x-2)^3}$. Turning point when

$$2 - \frac{2}{(x-2)^3} = 0$$

$$(x-2)^3 = 1$$

$$x - 2 = 1 \Rightarrow x = 3$$

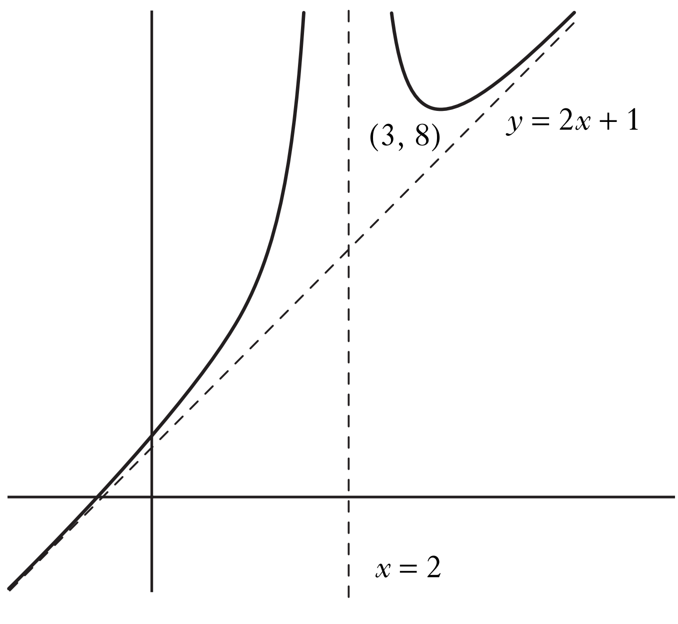
$$f''(x) = \frac{6}{(x-2)^4} > 0 \text{ for all } x.$$

The stationary point at (3, 8) is a minimum turning point. **[4]**

(d) $f(-2) = \dfrac{-16-28-8+5}{(-4)^2} < 0$; $f(0) = \dfrac{5}{4} > 0$.

Hence a root between –2 and 0. **[1]**

(e)



[2]

Marks

13. (*a*)

L_1: $x = 3 + 2s$; $y = -1 + 3s$; $z = 6 + s$

L_2: $x = 3 - t$; $y = 6 + 2t$; $z = 11 + 2t$

$\therefore$ for x: $3 + 2s = 3 - t \Rightarrow t = -2s$

$\therefore$ for y: $3s - 1 = 6 + 2t$

$7s = 7 \Rightarrow s = 1$; $t = -2$

$\therefore L_1$: $x = 5$; $y = 2$; $z = 6 + s = 7$

$\therefore L_2$: $x = 5$; $y = 2$; $z = 11 + 2t = 11 - 4 = 7$

ie L_1 and L_2 intersect at (5, 2, 7) **[6]**

(*b*)

$A(2,1,0)$; $B(3,3,-1)$; $C(5,0,2)$

$\vec{AB} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$; $\vec{AC} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

Equation of plane has form $3x - 5y - 7z = k$

$(2,1,0) \Rightarrow k = 1$

Equation is $3x - 5y - 7z = 1$. **[5]**

14. (*a*) $z^4 = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i\,(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

Hence the real part is $\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

The imaginary part is $(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

$= 4\cos\theta\sin\theta\,(\cos^2\theta - \sin^2\theta)$ **[5]**

(*b*) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ **[1]**

(*c*) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$ **[1]**

(*d*) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta\,(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

$= 8\,(\cos^4\theta - \cos^2\theta) + 1$

ie $k = 8$, $m = 4$, $n = 2$, $p = 1$. **[4]**

Marks

15. (*a*) $900 = A(15 - Q) + B(30 - Q)$
Letting $Q = 30$ gives $A = -60$
and $Q = 15$ gives $B = 60$

$$\frac{900}{(30-Q)(15-Q)} = \frac{-60}{(30-Q)} + \frac{60}{(15-Q)}$$ **[2]**

(*b*) $\frac{dQ}{dt} = \frac{(30-Q)(15-Q)}{900}$

$$\therefore \int \frac{900}{(30-Q)(15-Q)}\,dQ = \int dt$$

$$\therefore \int \frac{-60}{(30-Q)} + \frac{60}{(15-Q)}\,dQ = \int dt$$

$60\ln(30-Q) - 60\ln(15-Q) = t + C$

ie $60\ln\left(\frac{30-Q}{15-Q}\right) = t + C$

$A = 60$

$C = 60\ln 2 = 41{\cdot}59$ to 2 decimal places **[4]**

(i) $t = 60\ln\left(\frac{30-Q}{15-Q}\right) - 60\ln 2 = 60\ln\left(\frac{30-Q}{2(15-Q)}\right)$

When $Q = 5$, $t = 60\ln\frac{25}{20} = 13{\cdot}39$ minutes to 2 decimal places **[1]**

(ii) $\ln\left(\frac{30-Q}{2(15-Q)}\right) = \frac{t}{60}$

$30 - Q = 2(15-Q)e^{t/60}$

$Q(2e^{t/60} - 1) = 30(e^{t/60} - 1)$

$$Q = \frac{30(e^{t/60}-1)}{2e^{t/60}-1}$$

When $t = 45$, $Q = 10{\cdot}36$ grams to 2 decimal places. **[2]**

[*END OF SPECIMEN MARKING SOLUTIONS*]