C202/SQP252

Advanced Higher Applied Mathematics: Statistics

Time: 3 hours

Specimen Question Paper
for use in and after 2005

NATIONAL QUALIFICATIONS

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Statistics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**
4. A booklet of Statistical Formulae and Tables is supplied for all candidates.

SCOTTISH QUALIFICATIONS AUTHORITY

Section A (Statistics 1 and 2)

Marks

Answer all the questions.

1. A driver is considering buying a five year old Lafond car but knows that 30% of them have a defective gearbox. A mechanic offers to test drive any possible purchase in order to assess the gearbox. The mechanic reckons that he will correctly identify 90% of the defective gearboxes but admits there is a 20% chance that he will declare a sound gearbox to be defective.

   What is the probability that a gearbox assessed by the mechanic as sound is in fact sound? **5**

2. A type of lighting set for Christmas trees has 20 lights. The bulbs are selected at random from a population of bulbs which is 5% defective. Let X denote the number of defectives in a lighting set.

   (*a*) State the distribution of X and its parameters. **2**

   (*b*) Obtain the probability that a lighting set includes

   (i) no defective bulbs

   (ii) more than one defective bulb. **3**

3. A school has 200 male pupils and 300 female pupils. In order to gauge opinion on a proposed new school uniform the head teacher decides to take a sample of 50 pupils by selecting 20 males at random and 30 females at random.

   (*a*) State the type of sampling used and the probability that a particular pupil is selected in the sample. **2**

   (*b*) Explain the difference between this sampling method and simple random sampling. **2**

4. (*a*) By conducting a survey based on a random sample, a local community council surveyed opinion amongst residents on whether dogs should be admitted to a local park. Fifty-six per cent of those asked supported a ban. The 95% confidence limits for the proportion supporting the ban were 48% and 64%. State how you would respond to the claim that, on the basis of the opinion poll, the majority of residents were in favour of a ban. **2**

   (*b*) A shop selling telephones found that, during 1993, approximately 20% of customers chose the colour white. During the early weeks of 1994, a random sample of 200 sales indicated that 53 involved the choice of a white telephone.

   Use the normal approximation to the binomial distribution to test the shop manager's claim that the demand for white telephones has increased. **6**

Marks

5. The time required for a glass of beer to settle, following pouring, is known as surge time and is a quality characteristic monitored by brewers. Surge times (seconds) are given below for samples from two different brews.

Brew A	100	108	102	105	111			
Brew B	104	109	112	113	115	109	114	116

After ranking this data, the ranks for Brew A are 1, 2, 4, 5 and 8.

Perform an appropriate statistical test to assess whether or not the data provide any evidence that the brews have different median surge times. **6**

6. A biology student measured the length (cm) of a random sample of 8 leaves from a bush in the school grounds and recorded these results.

9·5	8·9	8·6	10·4	8·7	10·2	8·9	9·6

Find a 99% confidence interval for the mean leaf length of this bush.

Use this interval to comment on the hypothesis that the mean leaf length for this type of bush is 10 cm. **5**

7. (*a*) A correlation coefficient based on a random sample of 18 was computed to be 0·32. Can we conclude that the corresponding population correlation coefficient is greater than zero? **3**

(*b*) What is the minimum sample size necessary to conclude, at the 5% level, that a correlation coefficient of 0·32 differs significantly from zero? **2**

8. In a mechanical device, two components are assembled "end to end". The length of the first component is normally distributed with mean 10·00 mm and standard deviation 0·03 mm, while that of the second component is normally distributed with mean 20·00 mm and standard deviation 0·04 mm.

(*a*) Components are selected at random for assembly. Calculate the probability that overall length exceeds 30·10 mm. **4**

(*b*) Calculate 99% tolerance limits for overall length, ie limits between which 99% of overall lengths will lie. **3**

(*c*) Assembled components are fitted into a housing, selected at random, with length which is normally distributed with mean 30·20 mm and standard deviation 0·05 mm. Calculate the probability that the assembled components cannot be fitted into the housing. **3**

Marks

9. When a medical manufacturing process is in a state of statistical control, it produces tablets with weights which are normally distributed with mean 6·00 mg and standard deviation 0·13 mg.

Samples of 5 tablets are taken at regular intervals and the mean weight recorded. The means obtained during a day's production are given below.

Sample	1	2	3	4	5	6	7	8
Mean	6·04	5·89	5·99	6·06	5·96	6·08	5·87	6·25
Sample	9	10	11	12	13	14	15	16
Mean	6·05	5·86	5·96	6·08	6·19	6·05	5·92	5·90

(*a*) Calculate control limits for the sample mean weight of 5 tablets. **3**

(*b*) Plot a control chart for the manufacturing process and comment. **4**

(*c*) A false alarm occurs when a sample mean lies outside the control limits but there has been no change in μ and σ. Show that the probability of a false alarm is less than 0·005. **3**

10. In a city, traffic was observed during a randomly selected sample of 60 intervals, each of length 30 seconds.

The following table gives the frequency, *f*, with which *x* vehicles passed a point on a road.

Number of vehicles (x)	0	1	2	3	4	5	6	7	8
Number of intervals (f)	16	12	13	8	4	3	2	1	1

(*a*) Find the expected Poisson frequencies. **4**

(*b*) Carry out a χ^2 goodness-of-fit test to show that the Poisson distribution may not provide a good model of the situation. **5**

(*c*) Give a reason why the Poisson distribution may not be a good model for city traffic data. **1**

[*END OF SECTION A*]

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions.

11. Differentiate with respect to x

(*a*) $f(x) = (1 + 2x)\ \ln\ (1 + 2x),\ x > -\frac{1}{2}$, **3**

(*b*) $g(x) = e^{\cot 2x},\ 0 < x < \frac{\pi}{2}$. **2**

12. Expand

$$\left(x^2 - \frac{1}{x}\right)^3, \quad x \neq 0$$

and simplify as far as possible. **3**

13. (*a*) Obtain partial fractions for

$$\frac{x}{x^2 - 1}, \qquad x > 1.$$ **2**

(*b*) Use the result of (*a*) to find

$$\int \frac{x^3}{x^2 - 1}\, dx,\ \ x > 1.$$ **4**

14. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}.$$

Show that $AB = kI$ for some constant k, where I is the 3×3 identity matrix.
Hence obtain (i) the inverse matrix A^{-1}, and (ii) the matrix A^2B. **4**

15. Obtain

$$\int 2x\ \sin\ 4x\ dx.$$ **4**

Marks

16. A chemical plant food loses effectiveness at a rate proportional to the amount present in the soil. The amount M grams of plant food effective after t days satisfies the differential equation

$$\frac{dM}{dt} = kM, \text{ where } k \text{ is a constant.}$$

(*a*) Find the general solution for M in terms of t where the initial amount of plant food is M_0 grams. **3**

(*b*) Find the value of k if, after 30 days, only half the initial amount of plant food is effective. **3**

(*c*) What percentage of the original amount of plant food is effective after 35 days? **2**

(*d*) The plant food has to be renewed when its effectiveness falls below 25%. Is the manufacturer of the plant food justified in calling its product "sixty day super food"? **2**

[*END OF SECTION B*]

[*END OF SPECIMEN QUESTION PAPER*]

C202/SQP252

Advanced Higher
Applied Mathematics:
Statistics
Specimen Solutions

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Marks

Section A (Statistics 1 and 2)

1. P(def) = 0·3 P(diag | def) = 0·9 P(diag′ | def) = 0·1

P(ok) = 0·7 P(diag | ok) = 0·2 P(diag′ | ok) = 0·8

Bayes Theorem $P(\text{ok} \mid \text{diag}') = \dfrac{0\cdot 8\times 0\cdot 7}{0\cdot 8\times 0\cdot 7+0\cdot 1\times 0\cdot 3} = \dfrac{56}{59}$ **[5]**

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2. $B(20, 0\cdot 05)$ $P(B=0) = 0\cdot 358$ $P(B>1) = 1 - P(B\leq 1) = 0\cdot 264$ **[5]**

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3. Stratified random sampling with $p = 0\cdot 1$. **[2]**

Simple random sample gives every set the same chance of selection so that for example 50 boys could be selected—not possible in stratified. **[2]**

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4. (*a*) Since the confidence interval includes values below 50% I would disagree with the claim—we cannot be certain that there is a majority in favour of the ban. **[2]**

(*b*) $H_0 : \mu = 40$ $H_1 : \mu > 40$

1 tail test $B(200, 0\cdot 2)$

$$P(B \geq 53) \approx P\left(Z \geq \frac{53-\frac{1}{2}-40}{5\cdot 66}\right) = 0\cdot 014$$

Reject H_0 at 5%.

There is evidence to support the manager's claim. **[6]**

Marks

5. $H_0 : m_A = m_B$ $H_1 : m_A \neq m_B$ 2-tail test

Mann - Whitney $W_A = 20$ $W - \frac{1}{2}n(n+1) = 5$ $P(W \leq 5) = \frac{19}{1287} \approx 0{\cdot}015 < 0{\cdot}025$

Reject H_0 at 5% level ie there is evidence that the brews have different surge times. **[6]**

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6. $\bar{x} = 9{\cdot}35$ $s \approx 0{\cdot}687$ 99% CI is $\bar{x} \pm t\frac{\sigma}{\sqrt{n}} = 9{\cdot}35 \pm 3{\cdot}5\frac{0{\cdot}687}{\sqrt{8}}$

(n–1 divisor)

C.I. is $8{\cdot}50 \rightarrow 10{\cdot}2$ which only just contains 10 so it is just possible that the mean leaf length is 10 cm. **[5]**

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7. (*a*) $H_0 : \rho = 0$

$H_1 : \rho > 0$

$t = \frac{\sqrt{n-2}}{\sqrt{1-r^2}} \approx 1{\cdot}35$ $t_{5\%,16\text{df}} = 1{\cdot}75$ (one tail) no evidence that $\rho > 0$ **[3]**

(*b*) $n = 27$ $t \approx 1{\cdot}69$

$n = 28$ $t \approx 1{\cdot}72$

for each of these $t_{5\%} = 1{\cdot}71$ so that $n \geq 28$. **[2]**

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8. (*a*) $T = C_1 + C_2$ $T \sim N(30,\ 0{\cdot}5^2)$ $P(T > 30{\cdot}10) = P\left(Z > \frac{30{\cdot}1 - 30}{0{\cdot}05}\right) \approx 0{\cdot}023$ **[4]**

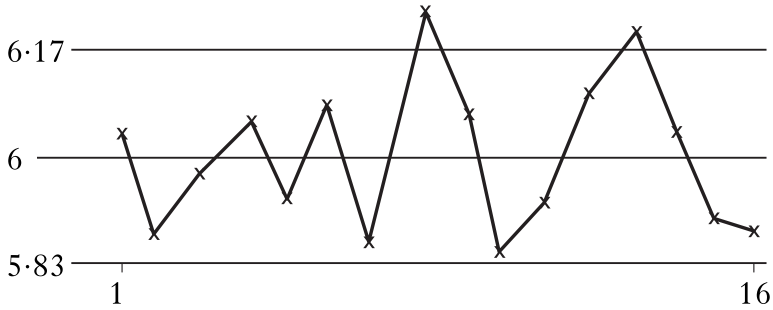
(*b*) $30 \pm 2{\cdot}58 \times 0{\cdot}05 = 30 \pm 0{\cdot}13$ **[3]**

(*c*) $(H - T) \sim N(0{\cdot}2, 0{\cdot}71^2)$ $P(H - T > 0) = P\left(Z > \frac{0 - 0{\cdot}2}{0{\cdot}071}\right) = 0{\cdot}998 \rightarrow p = 0{\cdot}002$ **[3]**

Marks

9. (*a*) $\mu \pm \frac{3\sigma}{\sqrt{n}} = 6 \pm 0\cdot17 = 5\cdot83 \rightarrow 6\cdot17$ **[3]**

(*b*)



There are 2 "out of control" points

[4]

(*c*) $P\left(\bar{X} > \mu + \frac{3\sigma}{\sqrt{n}}\right) = P(Z > 3) \approx 0\cdot01$

P (false alarm) $\approx 0\cdot002 < 0\cdot005$ **[3]**

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10. H_0 : Poisson is a good fit
H_1 : Data come from another distribution

(*a*) 8·5 16·7 16·2 10·6 5·1 2·0 0·7 0·2 0·0 **[4]**

(*b*) 8·5 16·7 16·2 10·6 8·0
16 12 13 8 11

$\chi^2 \approx 10$
$\chi^2_{5\%,\ 3df} \approx 7\cdot8$

Reject H_0 at 5% ie there is evidence that a Poisson model is inappropriate. **[5]**

(*c*) Random arrivals are unlikely in a city situation eg traffic lights, rush hours etc. **[1]**

Section B (Mathematics for Applied Mathematics)

Marks

11. (*a*) $f(x) = (1 + 2x) \ln (1 + 2x)$

$$f'(x) = 2 \ln (1 + 2x) + (1 + 2x) \frac{2}{1 + 2x}$$

$$= 2 \ln (1 + 2x) + 2$$ **[1,1,1]**

(*b*) $g(x) = e^{\cot 2x}$

$$g'(x) = -2 \operatorname{cosec}^2 2x \, e^{\cot 2x}$$ **[1, 1]**

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12. $$\left(x^2 - \frac{1}{x}\right)^3 = (x^2)^3 - 3(x^2)^2 \frac{1}{x} + 3x^2 \frac{1}{x^2} - \frac{1}{x^3}$$ **[1, 1]**

$$= x^6 - 3x^3 + 3 - \frac{1}{x^3}$$ **[1]**

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13. (*a*) $$\frac{x}{x^2 - 1} = \frac{A}{x-1} + \frac{B}{x+1}$$ **[1]**

$$= \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}$$ **[1]**

(*b*) $$\frac{x^3}{x^2 - 1} = x + \frac{x}{x^2 - 1}$$ **[1]**

$$= x + \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}.$$

$$\int \frac{x^3}{x^2-1}\,dx = \int x + \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}\,dx$$ **[1]**

$$= \tfrac{1}{2}x^2 + \tfrac{1}{2}\ln(x-1) + \tfrac{1}{2}\ln(x+1) + c$$

$$= \tfrac{1}{2}(x^2 + \ln(x^2 - 1)) + c$$ **[1, 1]**

Marks

14. $$AB = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2I$$ **[1]**

(i) $$A^{-1} = \frac{1}{2}B$$ **[1]**

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$ **[1]**

(ii) $$A^2B = A.AB = A.2I = 2A$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & -2 & -2 \end{pmatrix}$$ **[1]**

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15. $$\int 2x \sin 4x \, dx = 2x\int \sin 4x \, dx - \int\left(2\int \sin 4x \, dx\right)dx$$ **[1]**

$$= 2x.\tfrac{1}{4}(-\cos 4x) + \tfrac{1}{2}\int \cos 4x \, dx$$ **[1, 1]**

$$= -\tfrac{1}{2}x\cos 4x + \tfrac{1}{8}\sin 4x$$ **[1]**

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16. (*a*) *Marks*

$$\frac{dM}{dt} = kM$$

$$\int \frac{dM}{M} = \int k\, dt$$ **[1]**

$$\ln M = kt + c$$ **[1]**

$$t = 0,\ M = M_0 \Rightarrow c = \ln M_0$$

$$M = M_0 e^{kt}$$ **[1]**

(*b*) When $t = 30$, $M = \frac{1}{2} M_0$ so **[1]**

$$e^{30k} = 0 \cdot 5$$ **[1]**

$$k = \frac{1}{30} \ln 0 \cdot 5 \approx -0 \cdot 0231$$ **[1]**

(*c*) When $t = 35$,

$$\frac{M}{M_0} = e^{35k}$$ **[1]**

$$\approx 0 \cdot 4454 \approx 45\%$$ **[1]**

(*d*) When $\frac{M}{M_0} = 0 \cdot 25$

$$e^{kt} = 0 \cdot 25$$ **[1]**

$$t = \frac{1}{k} \ln 0 \cdot 25 = \frac{30 \ln 0 \cdot 25}{\ln 0 \cdot 5} = 60$$ **[1]**

The manufacturer is justified.

[*END OF SPECIMEN SOLUTIONS*]