AH

National
Qualifications
2023 MODIFIED

X847/77/12

Mathematics
Paper 2

THURSDAY, 4 MAY

10:30 AM – 12:30 PM

Total marks — 65

Attempt ALL questions.

You may use a calculator.

To earn full marks you must show your working in your answers.

State the units for your answer where appropriate.

You will not earn marks for answers obtained by readings from scale drawings.

Write your answers clearly in the spaces provided in the answer booklet. The size of the space provided for an answer is not an indication of how much to write. You do not need to use all the space.

Additional space for answers is provided at the end of the answer booklet. If you use this space you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

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FORMULAE LIST

Standard derivatives

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals

$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)+c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
e^{ax}	$\frac{1}{a}e^{ax}+c$

Summations

(Arithmetic series) $S_n = \frac{1}{2}n\left[2a+(n-1)d\right]$

(Geometric series) $S_n = \frac{a\left(1-r^n\right)}{1-r},\ r \neq 1$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \ldots$$

FORMULAE LIST (continued)

De Moivre's theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Vector product

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = \mathbf{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix} - \mathbf{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix} + \mathbf{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\end{aligned}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ, about the origin, $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

[Turn over

MARKS

Total marks — 65

Attempt ALL questions

1. The function f is defined by $f(x) = 2\sin^{-1} 3x$.
Find $f'(x)$. **2**

2. Find $\int \frac{x^2}{x^3 + 10}\,dx$. **2**

3. Matrix A is defined by $A = \begin{pmatrix} 2 & 2x & 4 \\ x & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix}$, where $x \in \mathbb{R}$.

(a) Find a simplified expression for the determinant of A. **2**

(b) Hence, determine whether A^{-1} exists for all values of x. **1**

4. Calculate the gradient of the tangent to the curve with equation $x^2 y^2 - 2y = \sin 3x$ at the point $(0, 0)$. **3**

5. (a) Write down and simplify the general term in the binomial expansion of $\left(3x - \frac{2}{x^2}\right)^8$. **3**

(b) Hence, or otherwise, determine the coefficient of x^{-1}. **2**

6. (a) Use the Euclidean algorithm to find d, the greatest common divisor of 703 and 399. **1**

(b) Find integers a and b such that $d = 703a + 399b$. **2**

(c) Hence find integers p and q such that $76 = 703p + 399q$. **1**

MARKS

7. (a) Solve the differential equation

$$\frac{dy}{dx} - 2y = 6e^{5x}$$

given that when $x = 0$, $y = -1$.

Express y in terms of x. **4**

(b) The solution of the differential equation in (a) is also a solution of

$$\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} = ke^{2x}, \; k \in \mathbb{R}.$$

Find the value of k. **2**

8. The fourth and seventh terms of a geometric sequence are 9 and 243 respectively.

(a) Find the:

(i) common ratio **1**

(ii) first term. **1**

(b) Show that $\frac{S_{2n}}{S_n} = 1 + 3^n$ where S_n represents the sum of the first n terms of this geometric sequence. **2**

9. Express 572_{10} in base 9. **2**

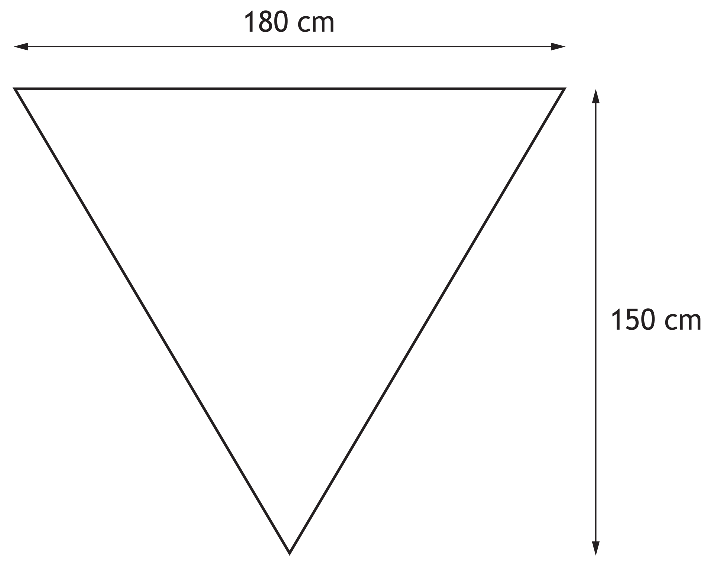
10. A curve is defined by $y = x^{5x^2}$, where $x > 0$.

Find $\frac{dy}{dx}$ in terms of x. **5**

[Turn over

11. On a building site, water is stored in a container.

The container is a cone with diameter 180 cm at its widest point and height of 150 cm. A cross section of the cone is shown below.



(a) Show that when the water level is at a height of h cm, $0 \le h \le 150$, the volume of water in the container can be written as

$$V = \frac{3\pi h^3}{25}.$$

[The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.] **1**

Water is pumped into the container at a constant rate of 10 litres per second.

(b) Find the rate at which the height is increasing when $h = 125$. **5**

12. Prove by induction that, for all positive integers n, $\sum_{r=1}^{n} 2^{r-1} r = 2^n(n-1)+1$. **5**

MARKS

13. Points scored in the long jump element of the decathlon can be calculated using a solution of the differential equation

$$(m-220)\frac{dP}{dm}=1.4P,\ m>220$$

where m is the distance jumped in centimetres and P the points scored.

Given that a jump of 807 centimetres scores 1079 points, find an expression for P in terms of m. **6**

14. A complex number is defined by $w = a + ib$, where a and b are positive real numbers.

Given $w^2 = 8 + 6i$, determine the values of a and b. **4**

15. A function $f(x)$ has the following properties:

- $f'(x)=\dfrac{x+1}{1+(x+1)^4}$
- the first term in the Maclaurin expansion of $f(x)$ is 1.

(a) Find the Maclaurin expansion of $f(x)$ up to and including the term in x^2. **3**

(b) Use the substitution $u=(x+1)^2$ to find $\displaystyle\int \frac{x+1}{1+(x+1)^4}\,dx$. **3**

(c) Determine an expression for $f(x)$. **2**

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