

X100/13/01

NATIONAL
QUALIFICATIONS
2015

WEDNESDAY, 20 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

- 1 Calculators may be used in this paper.
- 2 Candidates should answer **all** questions.
- 3 **Full credit will be given only where the solution contains appropriate working.**



Answer all the questions

1. Use the binomial theorem to expand and simplify

$$\left(\frac{x^2}{3} - \frac{2}{x}\right)^5.$$

4

2. (a) For $y = \frac{5x+1}{x^2+2}$, find $\frac{dy}{dx}$. Express your answer as a single, simplified fraction.

3

- (b) Given $f(x) = e^{2x}\sin^2 3x$, obtain $f'(x)$.

3

3. The sum of the first twenty terms of an arithmetic sequence is 320.

The twenty-first term is 37.

What is the sum of the first ten terms?

5

4. The equation $x^4 + y^4 + 9x - 6y = 14$ defines a curve passing through the point A(1, 2).

Obtain the equation of the tangent to the curve at A.

4

5. Obtain the value(s) of p for which the matrix $A = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$ is singular.

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6. For $y = 3^{x^2}$, obtain $\frac{dy}{dx}$.

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7. Use the Euclidean algorithm to find integers p and q such that

$$3066p + 713q = 1.$$

4

8. Given $x = \sqrt{t+1}$ and $y = \cot t$, $0 < t < \pi$,

obtain $\frac{dy}{dx}$ in terms of t .

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9. Show that

$$\binom{n+2}{3} - \binom{n}{3} = n^2,$$

for all integers, n , where $n \geq 3$.

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10. Obtain the exact value of $\int_0^2 x^2 e^{4x} dx$.

5

11. Write down the 2×2 matrix, M_1 , associated with a reflection in the y -axis.

Write down a second 2×2 matrix, M_2 , associated with an anticlockwise rotation through an angle of $\frac{\pi}{2}$ radians about the origin.

Find the 2×2 matrix, M_3 , associated with an anticlockwise rotation through $\frac{\pi}{2}$ radians about the origin followed by a reflection in the y -axis.

What single transformation is associated with M_3 ?

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12. Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8.

3

13. By writing z in the form $x + iy$:

(a) solve the equation $z^2 = |z|^2 - 4$;

3

(b) find the solutions to the equation $z^2 = i(|z|^2 - 4)$.

4

[Turn over

14. For some function, f , define

$$g(x) = f(x) + f(-x) \quad \text{and}$$

$$h(x) = f(x) - f(-x).$$

Show that $g(x)$ is an even function and that $h(x)$ is an odd function.

Hence show that $f(x)$ can be expressed as the sum of an even and an odd function. **4**

15. A line, L_1 , passes through the point $P(2, 4, 1)$ and is parallel to

$$\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

and a second line, L_2 , passes through $Q(-5, 2, 5)$ and is parallel to

$$\mathbf{u}_2 = -4\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

- (a) Write down the vector equations for L_1 and L_2 . **2**
- (b) Show that the lines L_1 and L_2 intersect and find the point of intersection. **4**
- (c) Determine the equation of the plane containing L_1 and L_2 . **4**

16. Solve the second order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x}$$

given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 0$. **10**

17. Find $\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)} dx$, $x > 3$. **9**

18. Vegetation can be irrigated by putting a small hole in the bottom of a cylindrical tank, so that the water leaks out slowly. Torricelli's Law states that the rate of change of volume, V , of water in the tank is proportional to the square root of the height, h , of the water above the hole.

This is given by the differential equation:

$$\frac{dV}{dt} = -k\sqrt{h}, \quad k > 0.$$

- (a) For a cylindrical tank with constant cross-sectional area, A , show that the rate of change of the height of the water in the tank is given by

$$\frac{dh}{dt} = \frac{-k}{A}\sqrt{h}. \quad 2$$

- (b) Initially, when the height of the water is 144 cm, the rate at which the height is changing is -0.3 cm/hr.

By solving the differential equation in part (a), show that $h = \left(12 - \frac{1}{80}t\right)^2$. 4

- (c) How many days will it take for the tank to empty? 2

- (d) Given that the tank has radius 20 cm, find the rate at which the water was being delivered to the vegetation (in cm^3/hr) at the end of the fourth day. 3

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