

# X100/13/01

---

NATIONAL  
QUALIFICATIONS  
2015

WEDNESDAY, 20 MAY  
1.00 PM – 4.00 PM

MATHEMATICS  
ADVANCED HIGHER

**Read carefully**

- 1 Calculators may be used in this paper.
- 2 Candidates should answer **all** questions.
- 3 **Full credit will be given only where the solution contains appropriate working.**



## Answer all the questions

1. Use the binomial theorem to expand and simplify

$$\left(\frac{x^2}{3} - \frac{2}{x}\right)^5.$$

4

2. (a) For  $y = \frac{5x+1}{x^2+2}$ , find  $\frac{dy}{dx}$ . Express your answer as a single, simplified fraction.

3

- (b) Given  $f(x) = e^{2x}\sin^2 3x$ , obtain  $f'(x)$ .

3

3. The sum of the first twenty terms of an arithmetic sequence is 320.

The twenty-first term is 37.

What is the sum of the first ten terms?

5

4. The equation  $x^4 + y^4 + 9x - 6y = 14$  defines a curve passing through the point A(1, 2).

Obtain the equation of the tangent to the curve at A.

4

5. Obtain the value(s) of  $p$  for which the matrix  $A = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$  is singular.

4

6. For  $y = 3^{x^2}$ , obtain  $\frac{dy}{dx}$ .

3

7. Use the Euclidean algorithm to find integers  $p$  and  $q$  such that

$$3066p + 713q = 1.$$

4

8. Given  $x = \sqrt{t+1}$  and  $y = \cot t$ ,  $0 < t < \pi$ ,

obtain  $\frac{dy}{dx}$  in terms of  $t$ .

3

9. Show that

$$\binom{n+2}{3} - \binom{n}{3} = n^2,$$

for all integers,  $n$ , where  $n \geq 3$ .

4

10. Obtain the exact value of  $\int_0^2 x^2 e^{4x} dx$ .

5

11. Write down the  $2 \times 2$  matrix,  $M_1$ , associated with a reflection in the  $y$ -axis.

Write down a second  $2 \times 2$  matrix,  $M_2$ , associated with an anticlockwise rotation through an angle of  $\frac{\pi}{2}$  radians about the origin.

Find the  $2 \times 2$  matrix,  $M_3$ , associated with an anticlockwise rotation through  $\frac{\pi}{2}$  radians about the origin followed by a reflection in the  $y$ -axis.

What single transformation is associated with  $M_3$ ?

4

12. Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8.

3

13. By writing  $z$  in the form  $x + iy$ :

(a) solve the equation  $z^2 = |z|^2 - 4$ ;

3

(b) find the solutions to the equation  $z^2 = i(|z|^2 - 4)$ .

4

[Turn over

14. For some function,  $f$ , define

$$g(x) = f(x) + f(-x) \quad \text{and}$$

$$h(x) = f(x) - f(-x).$$

Show that  $g(x)$  is an even function and that  $h(x)$  is an odd function.

Hence show that  $f(x)$  can be expressed as the sum of an even and an odd function. **4**

15. A line,  $L_1$ , passes through the point  $P(2, 4, 1)$  and is parallel to

$$\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

and a second line,  $L_2$ , passes through  $Q(-5, 2, 5)$  and is parallel to

$$\mathbf{u}_2 = -4\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

(a) Write down the vector equations for  $L_1$  and  $L_2$ . **2**

(b) Show that the lines  $L_1$  and  $L_2$  intersect and find the point of intersection. **4**

(c) Determine the equation of the plane containing  $L_1$  and  $L_2$ . **4**

16. Solve the second order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x}$$

given that when  $x = 0$ ,  $y = 1$  and  $\frac{dy}{dx} = 0$ . **10**

17. Find  $\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)} dx$ ,  $x > 3$ . **9**

18. Vegetation can be irrigated by putting a small hole in the bottom of a cylindrical tank, so that the water leaks out slowly. Torricelli's Law states that the rate of change of volume,  $V$ , of water in the tank is proportional to the square root of the height,  $h$ , of the water above the hole.

This is given by the differential equation:

$$\frac{dV}{dt} = -k\sqrt{h}, k > 0.$$

- (a) For a cylindrical tank with constant cross-sectional area,  $A$ , show that the rate of change of the height of the water in the tank is given by

$$\frac{dh}{dt} = \frac{-k}{A}\sqrt{h}. \quad 2$$

- (b) Initially, when the height of the water is 144 cm, the rate at which the height is changing is  $-0.3$  cm/hr.

By solving the differential equation in part (a), show that  $h = \left(12 - \frac{1}{80}t\right)^2$ . 4

- (c) How many days will it take for the tank to empty? 2

- (d) Given that the tank has radius 20 cm, find the rate at which the water was being delivered to the vegetation (in  $\text{cm}^3/\text{hr}$ ) at the end of the fourth day. 3

[END OF QUESTION PAPER]

**[BLANK PAGE]**

**DO NOT WRITE ON THIS PAGE**

**[BLANK PAGE]**

**DO NOT WRITE ON THIS PAGE**

**[BLANK PAGE]**

**DO NOT WRITE ON THIS PAGE**