2015 Applied Mathematics − Statistics

Advanced Higher

Finalised Marking Instructions

Part One: General Marking Principles for Applied Mathematics – Statistics – Advanced Higher

This information is provided to help you understand the general principles you must apply when marking candidate responses to questions in this Paper. These principles must be read in conjunction with the specific Marking Instructions for each question.

(a) Marks for each candidate response must always be assigned in line with these general marking principles and the specific Marking Instructions for the relevant question. If a specific candidate response does not seem to be covered by either the principles or detailed Marking Instructions, and you are uncertain how to assess it, you must seek guidance from your Team Leader/Principal Assessor.

(b) Marking should always be positive ie, marks should be awarded for what is correct and not deducted for errors or omissions.

GENERAL MARKING ADVICE: Applied Mathematics – Statistics – Advanced Higher

The marking schemes are written to assist in determining the "minimal acceptable answer" rather than listing every possible correct and incorrect answer. The following notes are offered to support Markers in making judgements on candidates' evidence, and apply to marking both end of unit assessments and course assessments.

These principles describe the approach taken when marking Advanced Higher Applied Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values/algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

Part Two: Marking Instructions for each Question

Section A

Question			Expected Answer(s)	Max Mark	Additional Guidance
A	**1**	**a**	Quota sampling	**1**	
A	**1**	**b**	120/8 = 15 so sample every 15th house Select a random integer, between 1 and 15 inclusive, as the number of the first house in the sample. Applying the same procedure with 125 houses means that none of the last five houses could be selected. Selection of a random integer, between 1 and 20 inclusive, as the number of the first house in the sample would mean that all houses had a chance of being included in the sample.	**4**	Other ideas are acceptable eg change the sample size to 5 (or 25) and sample every 25th (or 5th) or take a simple random sample instead
A	**2**	**a**	$\mathrm{P}\left(\mathrm{F}_1 \mid \text{Sample of n includes } x \text{ defective}\right)$ $= \dfrac{\mathrm{P}\left(\text{Sample of n includes } x \text{ defective} \cap \mathrm{F}_1\right)}{\mathrm{P}\left(\text{Sample of n includes } x \text{ defective}\right)}$ $= \dfrac{\mathrm{P}\left(\mathrm{F}_1 \cap \text{Sample of n includes } x \text{ defective}\right)}{\mathrm{P}\left(\text{Sample of n includes } x \text{ defective}\right)}$ $= \dfrac{\mathrm{P}\left(\mathrm{F}_1\right)\mathrm{P}\left(\text{Sample of n includes } x \text{ defective} \mid \mathrm{F}_1\right)}{\mathrm{P}\left(\text{Sample of n includes } x \text{ defective}\right)}$ $= \dfrac{0 \cdot 5\binom{n}{x} p_1^x \left(1-p_1\right)^{n-x}}{0 \cdot 5\binom{n}{x} p_1^x \left(1-p_1\right)^{n-x} + 0 \cdot 5\binom{n}{x} p_2^x \left(1-p_2\right)^{n-x}}$ $= \dfrac{p_1^x \left(1-p_1\right)^{n-x}}{p_1^x \left(1-p_1\right)^{n-x} + p_2 \left(1-p_2\right)^{n-x}}$	**4**	The 0·5 and $^{n}C_{x}$ have to be carefully explained Other methods (eg tree or Venn diagram) are acceptable
A	**2**	**b**	When $p_1 = p_2$ the probability is 0·5 indicating that the batch is equally likely to have come from either factory	**1**	'comment' is one step further than 'calculate' so 0·5 is not sufficient

Question			Expected Answer(s)	Max Mark	Additional Guidance
A	**3**	**a**	$\overline{X} \approx \mathrm{N}\left(5, \frac{25}{16}\right)$ $\mathrm{P}\left(4 < \overline{X} < 6\right)$ $\approx \mathrm{P}\left(\frac{4-5}{1 \cdot 25} < Z < \frac{6-5}{1 \cdot 25}\right)$ $= \mathrm{P}\left(-0 \cdot 8 < Z < 0 \cdot 8\right)$ $= 0 \cdot 5763$	**3**	a continuity correction is not appropriate here
A	**3**	**b**	$\mathrm{P}\left(4 < \overline{X} < 6\right)$ $\approx \mathrm{P}\left(\frac{4-5}{0 \cdot 5} < Z < \frac{6-5}{0 \cdot 5}\right)$ $= \mathrm{P}\left(-2 < z < 2\right)$ $= 0 \cdot 9545$	**2**	nor here
A	**3**	**c**	The distribution of the sample mean is more closely approximated by a normal distribution as the sample size increases	**1**	detailed statement required
A	**4**	**a**	There are 12 positive differences and 8 negatives. Under the null hypothesis $X \sim \mathrm{B}\left(20, 0 \cdot 5\right)$ and $\mathrm{P}\left(X \leq 8\right) = 0 \cdot 2517$ The p-value is thus 0·5034 > 0·05 and so there is no evidence that the **median** difference differs from zero	**4**	
A	**4**	**b**	Since the given p-value is less than 0·05 there is evidence that the **mean** difference differs significantly from zero. The conclusion is unsafe as use of the t-test requires the underlying distribution to be normal. The highly skewed histogram suggests otherwise.	**2**	test for a mean, rather than median, difference being zero
A	**4**	**c**	The null hypotheses cannot be rejected for either the sign test that the median difference in activity is zero or the t-test that the mean difference of the transformed activities is zero so neither test provides evidence that the fungicide affects enzyme activity.	**2**	

Question			Expected Answer(s)	Max Mark	Additional Guidance
A	**5**	**a**	A 95% confidence interval is given by $\bar{x} \pm 1\cdot96\frac{\sigma}{\sqrt{n}}$ $= \bar{x} \pm 1\cdot96\frac{2}{\sqrt{16}}$ $= \bar{x} \pm 0\cdot98 \approx \bar{x} \pm 1$	**2**	a *t*-statistic is not appropriate here
A	**5**	**b**	The probability that a 95% confidence interval captures the true mean is 0·95 so the probability that all 20 capture the true mean is $0\cdot95^{20} = 0\cdot358$	**2**	
A	**5**	**c**	A 50% confidence interval is given by:- $\bar{x} \pm 0\cdot67\frac{\sigma}{\sqrt{n}}$ $= \bar{x} \pm 0\cdot67\frac{2}{\sqrt{16}}$ $= \bar{x} \pm 0\cdot34$ This is a narrower interval so there would be a higher probability that some 50% confidence intervals would fail to capture the mean.	**3**	detailed statement required
A	**6**		$H_0 : \rho = 0 \qquad H_1 : \rho \neq 0$ $t = \frac{r}{\sqrt{\frac{1-r^2}{n-2}}} = \frac{0\cdot395}{\sqrt{\frac{1-0\cdot395^2}{40-2}}}$ $= 2\cdot65$ with 38 df The critical regions are $\lvert t \rvert > 2\cdot712(1\%)$ and $\lvert t \rvert > 2\cdot024(5\%)$ and 2·65 lies within only the 5% critical region so we can only reject the null hypothesis at the 5% level.	**6**	hypotheses need not be stated

<table>
<tr><th colspan="3">Question</th><th>Expected Answer(s)</th><th>Max Mark</th><th>Additional Guidance</th></tr>
<tr><td>A</td><td>7</td><td>a</td><td>
Expected frequency of 0 hurricanes is given by
$25\times\frac{1\cdot52^0}{0!}\,e^{-1\cdot52}=5\cdot468$
Expected frequency of 3 or more hurricanes is given by
$25-\left(5\cdot468+8\cdot311+6\cdot316\right)=4\cdot905$.
<table>
<tr><th>Observed frequency</th><th>Expected frequency</th><th>Contribution to chi-squared</th></tr>
<tr><td>3</td><td>5·468</td><td>1·114</td></tr>
<tr><td>10</td><td>8·311</td><td>0·343</td></tr>
<tr><td>8</td><td>6·316</td><td>0·449</td></tr>
<tr><td>4</td><td>4·905</td><td>0·167</td></tr>
</table>
The test statistic is 2·073 which follows a chi-squared distribution with 3 degrees of freedom and a 5% critical value of 7·815.

Since 2·073 < 7·815 the hypothesis that the number of major hurricanes per annum has a Poisson (1·52) distribution cannot be rejected. We may conclude that there is no evidence against the theory that major hurricanes occur randomly in time.
</td><td>6</td><td>
3df since a mean given

It would certainly be acceptable to reduce to 3 categories with 2 df since the final E_i is very close to 5
</td></tr>
<tr><td>A</td><td>7</td><td>b</td><td>
The number of hurricanes per 5-year period would have the Poisson distribution with mean 5 × 1·52 = 7·6.
$X \sim \text{Poi}\left(7\cdot6\right)$
$\text{P}\left(X\geq17\right)=1-\text{P}\left(X\leq16\right)=0\cdot0022$
Since this probability is small there is some evidence that the rate of occurrence of major hurricanes has increased.
</td><td>3</td><td>
It would be acceptable to use Tables with an approximate mean of 7·5, yielding 0·0020 or to get 0·0023 by linear interpolation.

A final comment on the rate of occurrence of hurricanes is a requirement
</td></tr>
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Question			Expected Answer(s)	Max Mark	Additional Guidance
A	**8**	**a**	All points lie between the 3-sigma limits	**1**	
A	**8**	**b**	$\mu + 3\frac{\sigma}{\sqrt{n}} = 7\cdot541$ $\sigma = \frac{(7\cdot541 - \mu)\sqrt{n}}{3}$ $= 0\cdot035$ $\mathrm{P}(7\cdot400 < X < 7\cdot600)$ $= \mathrm{P}\left(\frac{7\cdot400 - 7\cdot520}{0\cdot035} < Z < \frac{7\cdot600 - 7\cdot520}{0\cdot035}\right)$ $= \mathrm{P}(-3\cdot43 < Z < 2\cdot29)$ $= 0\cdot9886$	**5**	
A	**8**	**c**	Adjust the process mean to 7·500, the mid-point of the specification interval. $\mathrm{P}\left(\frac{7\cdot600 - 7\cdot500}{0\cdot035} < Z < \frac{7\cdot400 - 7\cdot500}{0\cdot035}\right)$ $= \mathrm{P}(-2\cdot86 < Z < 2\cdot86)$ $= 0\cdot9958$	**3**	
A	**8**	**d**	In order to increase the proportion still further the process variability would have to be reduced.	**1**	

Question			Expected Answer(s)	Max Mark	Additional Guidance
A	**9**	**a**	$S_{xx} = 685 - \frac{13^2}{19} = 676 \cdot 1053$ $t = \frac{b}{s / \sqrt{S_{xx}}}$ $= \frac{-0 \cdot 06029}{0 \cdot 2315 / \sqrt{676 \cdot 1053}}$ $= -6 \cdot 77$ $< -3 \cdot 965$ (17df, 2-tail, 0·1% level) yielding very strong evidence that the slope parameter differs from zero	**4**	
A	**9**	**b**	The predicted winning time is given by $y = 11 \cdot 26 - 0 \cdot 06029 \times 11 = 10 \cdot 60$ The model predicts that the time for the 100m race will drop to zero in the long term which is clearly not realistic.	**2**	
A	**9**	**c**	For 2008, $x = 10$ Residual = Data − Fit $= 10 \cdot 78 - (11 \cdot 26 - 0 \cdot 06029 \times 10)$ $= 0 \cdot 1229$ The rough U shape suggests that a non-linear model might be appropriate.	**3**	detailed statement required
A	**9**	**d**	The exponential term tends to zero, so this model does not predict a time of zero in the future. 10·604 is the predicted time in the long term.	**3**	A mathematical treatment is required here for full credit

[*END OF SECTION A*]

Section B (Mathematics for Applied Mathematics)

Question			Expected Answer(s)	Max Mark	Additional Guidance
B	**1**		$y = e^{5x}\tan 2x$ $\frac{dy}{dx} = e^{5x}.\frac{d}{dx}(\tan 2x) + \tan 2x.\frac{d}{dx}(e^{5x})$ $= e^{5x}.2\sec^2 2x + \tan 2x.5e^{5x}$ $= e^{5x}(2\sec^2 2x + 5\tan 2x)$	**3**	1: form of product rule 1: one derivative correct 1: other derivative correct (Factorisation not needed)
B	**2**	**a**	$A^2 = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}$ $\det A^2 = (4\times -4) - (2\times -10) = 4$ Since $\det A^2 \neq 0$, inverse of A^2 exists	**2**	1: Matrix A^2 correct 1: correct reason stated
B	**2**	**b**	$A^2B = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix}$ Inverse of $A^2 = \frac{1}{4}\begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$ Pre-multiply by $(A^2)^{-1}$ $IB = \frac{1}{4}\begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -11 \\ 0 & -5 \end{pmatrix}$ **<u>ALTERNATIVE SOLUTION</u>** Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A^2B = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix}$ $4a - 10c = 4$ $\quad$ $4b - 10d = 6$ $2a - 4c = 2$ $\quad$ $2b - 4d = -2$ Hence, $a = 1$ $\quad$ $b = -11$ $c = 0$ $\quad$ $d = -5$	**3**	1: Statement of inverse A^2 1: multiplying both sides by $(A^2)^{-1}$ 1: matrix B 1: Simultaneous equations 1: Two solutions 1: Remaining two solutions.

Question			Expected Answer(s)	Max Mark	Additional Guidance
B	**3**		$y=\dfrac{\sin x}{2-\cos x}$ $\dfrac{dy}{dx}=\dfrac{(2-\cos x).\cos x-\sin x(\sin x)}{(2-\cos x)^2}$ $=\dfrac{2\cos x-(\cos^2 x+\sin^2 x)}{(2-\cos x)^2}$ $=\dfrac{2\cos x-1}{(2-\cos x)^2}$ For a S.P., $\dfrac{dy}{dx}=0 \Leftrightarrow \dfrac{2\cos x-1}{(2-\cos x)^2}=0$ $\Leftrightarrow 2\cos x-1=0$ $\Leftrightarrow \cos x=\dfrac{1}{2}$ $x=\dfrac{\pi}{3}$ when $x=\dfrac{\pi}{3}$, $y=\dfrac{\sin\frac{\pi}{3}}{\left(2-\cos\frac{\pi}{3}\right)}=\dfrac{\sqrt{3}}{3}$	**5**	1: form of quotient rule with substitution or product rule 1: derivative 1: Use o $\sin^2 x+\cos^2 x=1$ to simplify expression 1: *x* coordinate 1: *y* coordinate
B	**4**		$\log_a 2+\log_a 4+\log_a 8=6\log_a 2$ $\sum_{r=1}^{100}\log_a 2^r=\log_a 2+\log_a 2^2+\log_a 2^3+...+\log_a 2^{100}$ $=\log_a 2(1+2+3+....+100)$ $=\log_a 2\left(\dfrac{100(101)}{2}\right)$ $= 5050\log_a 2$	**4**	1: Staement of answer 1: Expansion 1: simplification of indices ***and*** factorising 1: correct answer

Question			Expected Answer(s)	Max Mark	Additional Guidance
B	**5**		$\frac{1}{\cos x}\frac{dy}{dx}+y\tan x=\tan x$	**6**	
			$\cos x\times\frac{1}{\cos x}\frac{dy}{dx}+\cos x.y\tan x=\cos x.\tan x$		1: Multiply by cos*x*,
			* $\frac{dy}{dx}+y\sin x=\sin x$		1: form of I.F.
			Integrating Factor is $e^{\int\sin x dx}$ $= e^{-\cos x}$		1: I.F
			$e^{-\cos x}.\frac{dy}{dx}+e^{-\cos x}.y\sin x=e^{-\cos x}.\sin x$		
			$\frac{d}{dx}\left(y.e^{-\cos x}\right)=e^{-\cos x}.\sin x$		1: expressing LHS as correct exact differential
			Integrate both sides,		
			$\int\frac{d}{dx}\left(y.e^{-\cos x}\right)dx=\int e^{-\cos x}.\sin x\ dx$		
			$ye^{-\cos x} = e^{-\cos x} + C$		1: Integrating RHS
			$y=1+\frac{C}{e^{-\cos x}}$		1: Explicit function for *y*
			General Solution $y = 1 + Ce^{\cos x}$		
			<u>ALTERNATIVE SOLUTION</u> – From *		
			$\frac{dy}{dx}=\sin x(1-y)$		
			$\int\frac{dy}{1-y}=\int\sin x dx$		1: Separate variables
			$-\ell n\|1-y\|=-\cos x+C$		1: Integrate both sides,
			$e^{-\ell n\|1-y\|}=e^{-\cos x+C}$		1: Take exponential of both sides
			$\Leftrightarrow\frac{1}{1-y}=Ae^{-\cos x}$		1: Algebra of exponentials
			$\Leftrightarrow 1-y=Be^{\cos x}$		
			$\Leftrightarrow y=1-Be^{\cos x}$		1: Explicit function for *y*

Question			Expected Answer(s)	Max Mark	Additional Guidance
B	**6**	**a**	$\frac{1}{1-y^2}=\frac{1}{(1+y)(1-y)}=\frac{A}{1+y}+\frac{B}{1-y}$ $1 = A(1-y) + B(1+y)$ $A=\frac{1}{2}$ $B=\frac{1}{2}$ $\frac{1}{1-y^2}=\frac{1}{2}\left(\frac{1}{1+y}+\frac{1}{1-y}\right)$	**3**	1: form of partial fractions 1: constant value *A* 1: constant value *B*
B	**6**	**b**	**Substitution integral:** $u=\sqrt{1-x}$ $\frac{du}{dx}=\frac{1}{2}(1-x)^{-\frac{1}{2}}\times -1$ $=\frac{-1}{2\sqrt{1-x}}$ $-2du=\frac{dx}{\sqrt{1-x}}$ **Using** $u=\sqrt{1-x}$ $x=1-u^2$ $\int\frac{dx}{x\sqrt{1-x}}$ $=\int\frac{-2du}{x}$ $=-2\int\frac{du}{1-u^2}$ $=-2\int\frac{1}{2}\left(\frac{1}{1+u}+\frac{1}{1-u}\right)du$ $=-\left(\ell n\lvert 1+u\rvert-\ell n\lvert 1-u\rvert\right)+C$ $=\ell n\left\lvert 1-\sqrt{1-x}\right\rvert-\ell n\left\lvert 1+\sqrt{1-x}\right\rvert+C$ $=\ell n\left\lvert\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right\rvert+C$	**6**	1: correct derivative 1: express *x* in terms of *u* 1: replace all terms 1: use of partial fractions 1: integration 1: replace all *u* terms (do not penalise omission of + C or moduli signs)

[END OF SECTION B**]**

[END OF MARKING INSTRUCTIONS**]**