National
Qualifications
2021 ASSESSMENT RESOURCE

X802/77/11

Mathematics of Mechanics

Duration — 2 hours 50 minutes

Total marks — 90

SECTION 1 — 80 marks

Attempt ALL questions.

SECTION 2 — 10 marks

Attempt EITHER Part A OR Part B.

You may use a calculator.

To earn full marks you must show your working in your answers.

State the units for your answer where appropriate. Any rounded answer should be accurate to an appropriate number of significant figures unless otherwise stated.

Write your answers clearly in the spaces provided in the answer booklet. The size of the space provided for an answer is not an indication of how much to write. You do not need to use all the space.

Additional space for answers is provided at the end of the answer booklet. If you use this space you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

FORMULAE LIST

Newton's inverse square law of gravitation

$$F = \frac{GMm}{r^2}$$

Simple harmonic motion

$$v^2 = \omega^2(a^2 - x^2)$$

$$x = a\sin(\omega t + \alpha)$$

Centre of mass

Triangle: $\frac{2}{3}$ along median from vertex.

Semicircle: $\frac{4r}{3\pi}$ along the axis of symmetry from the diameter.

Coordinates of the centre of mass of a uniform lamina, area A square units, bounded by the equation $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\bar{x} = \frac{1}{A}\int_a^b xy\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}y^2\,dx$$

Standard derivatives	
$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax) + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$

MARKS

SECTION 1 — 80 marks

Attempt ALL questions

Note that g m s^{-2} denotes the magnitude of the acceleration due to gravity. Where appropriate, take its magnitude to be 9·8 m s^{-2}.

1. Two rail trucks are travelling towards each other in opposite directions on the same straight, smooth horizontal track.

One truck of mass m kg is moving at 0·75 m s^{-1} and the second truck of mass 400 kg is moving at 0·2 m s^{-1}.

When the trucks collide they become attached and continue moving together at 0·37 m s^{-1} in the direction of the first truck.

Calculate the value of m. **3**

2. A function is defined by $f(x) = e^{3x} \sin 2x$.

Find $f'\left(\frac{\pi}{4}\right)$. **3**

3. A particle of mass m kg hangs in equilibrium at the centre of a light inextensible string of length 3 metres, fixed between two horizontal points that are 2 metres apart.

The string makes an angle of θ with the horizontal.

(a) Find an expression for the tension in the string in terms of m and g. **3**

(b) The distance between the two fixed points increases but does not exceed 3 metres.

Explain, with reference to θ, what happens to the tension in the string. **1**

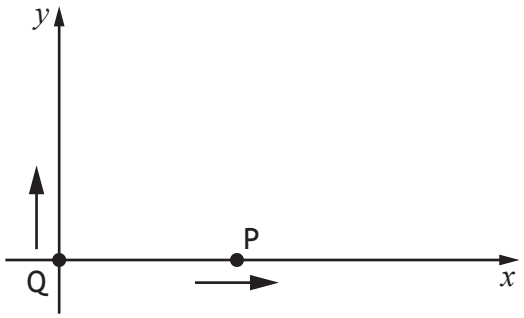
[Turn over

MARKS

4. A particle P is moving along the x-axis such that its displacement in metres relative to an origin is given by $\mathbf{s}_P = \left(\frac{1}{3}t^3 + \frac{1}{2}t^2 + 2t + 1\right)\mathbf{i}$, where t is the time measured in seconds, $t \geq 0$.

A second particle Q is moving along the y-axis with a constant acceleration of $3\mathbf{j}\,\text{m s}^{-2}$.

Q is initially at the origin and moving with a velocity of $5\mathbf{j}\,\text{m s}^{-1}$.



(a) Find the time when the particles have the same speed. **4**

(b) Calculate the distance between the particles at this time. **2**

5. Find the exact value of the integral $\int_0^1 3x\sqrt{4-3x^2}\,dx$ using the substitution $u = 4 - 3x^2$. **4**

6. A particle is projected vertically upwards.

At the same instant a second particle, vertically above it, is released from rest.

The particles collide after 3 seconds. At this time, the first particle is still moving upwards and the particles are travelling with equal speeds.

(a) Find the initial speed of the first particle. **3**

(b) Determine how far apart the particles are initially. **2**

MARKS

7. A particle of mass 0·5 kg is projected from a point A up the line of greatest slope of a rough plane inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$.

The coefficient of friction between the particle and the plane is $\frac{5}{8}$.

(a) The particle comes to rest instantaneously at a point B on the plane, where the distance AB = 3·5 metres.

(i) Find the work done against friction as the particle moves from point A to point B. **3**

(ii) Find the speed of projection of the particle from point A. **3**

The particle then begins to accelerate down the slope.

(b) Find the speed of the particle when it returns to point A. **2**

8. Relative to a suitable set of axes, the motion of a particle after t seconds is described by the parametric equations

$$x = t\sin 2t,\ y = \cos 3t$$

where x and y are distances in metres.

(a) Determine the direction of motion of the particle relative to the x-axis after $\frac{\pi}{6}$ seconds. **4**

(b) Calculate the instantaneous speed of the particle at this time. **1**

9. Two small boats, A and B, are each sailing at constant velocities in calm water.

A has a velocity of $\begin{pmatrix} 14 \\ 6 \end{pmatrix}$ km h^{-1} while B has a velocity of $\begin{pmatrix} 10 \\ -4 \end{pmatrix}$ km h^{-1}, where the x and y components are in the east/west and north/south directions respectively.

Initially B is 6 km due north of A.

(a) (i) Calculate the time it takes until the boats are closest to each other. **5**

(ii) Calculate the distance between A and B at this time. **1**

(b) Determine how many more minutes it takes until A is due east of B. **2**

[Turn over

MARKS

10. Find the particular solution of the differential equation

$$\frac{dy}{dx} - 2xy = \frac{e^{x^2}}{(x+1)^2}, \quad x > -1$$

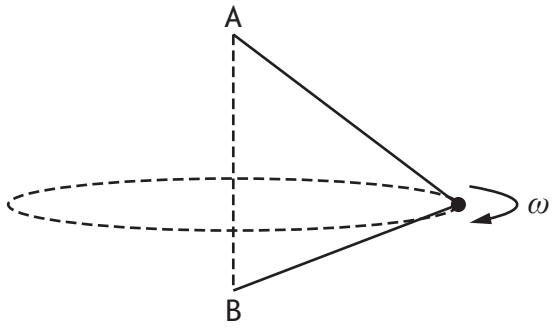
given that $y = 5$ when $x = 0$.

Express your answer in the form $y = f(x)$. **6**

11. A bead of mass m is attached to two fixed points A and B by two light inextensible strings.

A is 14 cm vertically above B and the string connecting the bead to A is 15 cm long.

The bead moves in a horizontal circle whose centre is 9 cm vertically below A.



The tension in the string to A is four times the tension in the string to B and the bead moves with constant angular velocity ω rad s^{-1}.

Calculate the value of ω. **6**

12. A curve is defined by the equation $3x^2 + 2xy - y^2 + 12 = 0$.

Use implicit differentiation to show that, at stationary points,

$$3 + (x - y)\frac{d^2y}{dx^2} = 0.$$ **5**

MARKS

13. A skydiver of mass m kg begins their freefall from rest.

They experience resistances of magnitude mkv^2 where k is a constant and v m s^{-1} is the velocity of the skydiver at any point during the descent.

(a) Show that the velocity can be expressed as

$$v = \sqrt{\frac{g\left(1 - e^{-2kx}\right)}{k}}$$

where x metres is the distance fallen by the skydiver. **6**

When the forces are in equilibrium it can be assumed that the skydiver is falling at a maximum velocity of 53 m s^{-1}.

(b) Find the value of k. **2**

(c) Calculate the distance fallen by the skydiver when they reach 99% of this maximum velocity of 53 m s^{-1}. **2**

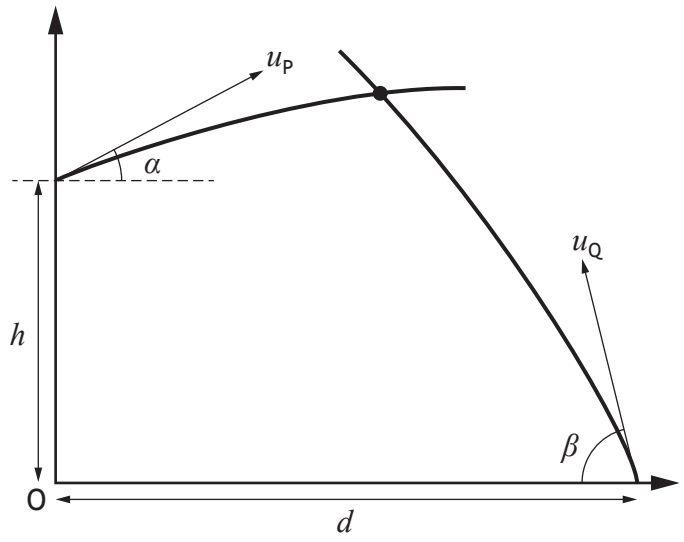
[Turn over

MARKS

14. Two particles, P and Q, are projected in the same vertical plane.

Relative to the origin, particle P is projected from point $h\,\mathbf{j}$ and particle Q is projected from point $d\,\mathbf{i}$, where $\mathbf{i}$ and $\mathbf{j}$ represent unit vectors in the horizontal and vertical directions respectively.

The particles are projected at angles α and β to the horizontal, as shown in the diagram, with speeds u_P and u_Q respectively.



(a) The particles collide after t seconds. Show clearly that $h = t\left(u_Q \sin\beta - u_P \sin\alpha\right)$ and $d = t\left(u_P \cos\alpha + u_Q \cos\beta\right)$. **5**

(b) Given that $d = 3h$, $\alpha = 20°$ and $\beta = 70°$, find the value of $\dfrac{u_P}{u_Q}$. **2**

[END OF SECTION 1]

MARKS

SECTION 2 — 10 marks

Attempt EITHER Part A OR Part B

Part A

15. A particle P is moving horizontally with simple harmonic motion. Its displacement from a fixed point O is given by $x = 8\sin\left(\frac{\pi}{3}t + \alpha\right)$ where x is measured in metres and t is measured in seconds. Initially P is 4 metres from O.

(a) (i) Calculate the period of the motion. **1**

(ii) Find the value of α. **1**

(b) Find when P is at its maximum displacement from O for the first time. **2**

16. A mass m kg is suspended from a point A by a string of natural length l metres and modulus of elasticity 4 mg. When the mass is in equilibrium, it is at a point B.

(a) Find an expression, in terms of l, for the distance AB. **2**

The mass is now pulled down a distance such that, when it is released, the subsequent motion is simple harmonic.

(b) (i) Show that $\ddot{x} = -\frac{4g}{l}x$, where x metres is the displacement from the equilibrium position. **2**

(ii) Calculate the maximum acceleration of the mass when the amplitude of the motion is $\frac{l}{5}$ metres. **2**

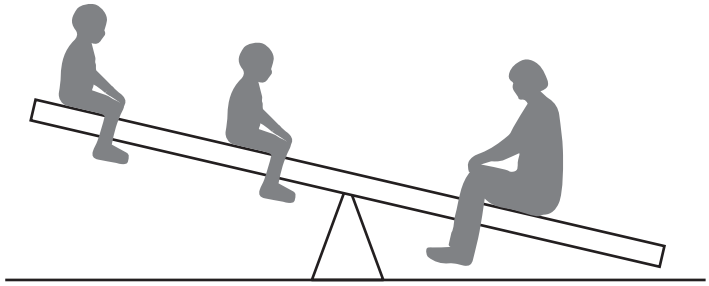
[Turn over

MARKS

Part B

17. Two children, each of mass 16 kg, sit on the same side of a uniform seesaw. They sit 2 m and 0·5 m from the centre support.

An adult of mass 80 kg sits x metres from the centre support on the opposite side of the seesaw. The overall turning effect is 20 g N m clockwise about the centre.

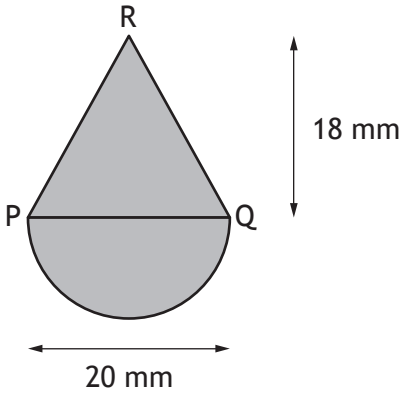


(a) Calculate the value of x. **2**

The adult moves to produce a turning effect of 20 g N m anticlockwise.

(b) Calculate the new distance of the adult from the centre support. **2**

18. An earring is a uniform lamina consisting of a semicircle of diameter PQ = 20 mm attached to an isosceles triangle PQR of height 18 mm.



(a) Find the distance of the centre of mass of the earring from the mid-point of PQ. **4**

The manufacturer wants to create an earring whose centre of mass lies on the line PQ. To achieve this they cut a circular hole of radius 5 mm, centred on the axis of symmetry of the earring.

(b) Calculate the distance between the centre of the circle and the mid-point of PQ. **2**

[END OF SECTION 2]

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