



National
Qualifications
2018

X747/77/11

Mathematics

THURSDAY, 3 MAY

INSTRUCTIONS TO CANDIDATES

Candidates should enter their surname, forename(s), date of birth, Scottish candidate number and the name and Level of the subject at the top of their first answer sheet.

Total marks — 100

Attempt ALL questions.

You may use a calculator.

Full credit will be given only to solutions which contain appropriate working.

State the units for your answer where appropriate.

Answers obtained by readings from scale drawings will not receive any credit.

Write your answers clearly on your answer sheet.

Questions marked with an asterisk differ in some respects from those in the printed paper.

Marks are shown in square brackets at the end of each question or part question.

An OW in the margin indicates a new question.

A separate formulae sheet is provided.

FORMULAE LIST

| Standard derivatives | |
|--------------------------|----------------------------------|
| $f(x)$ | $f'(x)$ |
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\tan^{-1} x$ | $\frac{1}{1+x^2}$ |
| $\tan x$ | $\sec^2 x$ |
| $\cot x$ | $-\operatorname{cosec}^2 x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cot x$ |
| $\ln x$ | $\frac{1}{x}$ |
| e^x | e^x |

| Standard integrals | |
|----------------------------|---|
| $f(x)$ | $\int f(x) dx$ |
| $\sec^2(ax)$ | $\frac{1}{a} \tan(ax) + c$ |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\left(\frac{x}{a}\right) + c$ |
| $\frac{1}{a^2+x^2}$ | $\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$ |
| $\frac{1}{x}$ | $\ln x + c$ |
| e^{ax} | $\frac{1}{a} e^{ax} + c$ |

Summations

(Arithmetic series)
$$S_n = \frac{1}{2}n[2a + (n-1)d]$$

(Geometric series)
$$S_n = \frac{a(1-r^n)}{1-r}, r \neq 1$$

$$\sum_{r=1}^n r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^n r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r \quad \text{where} \quad \binom{n}{r} = {}^n C_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \dots$$

FORMULAE LIST (continued)

De Moivre's theorem

$$[r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta)$$

Vector product

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin \theta \hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ , about the origin, $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

Total marks – 100
Attempt ALL questions

1. (a) Given $f(x) = \sin^{-1} 3x$, find $f'(x)$. [2 marks]
- (b) Differentiate $y = \frac{e^{5x}}{7x+1}$. [2 marks]
- (c) For $y \cos x + y^2 = 6x$, use implicit differentiation to find $\frac{dy}{dx}$. [4 marks]
2. Use partial fractions to find $\int \frac{3x-7}{x^2-2x-15} dx$. [4 marks]
3. (a) Write down and simplify the general term in the binomial expansion of $\left(2x + \frac{5}{x^2}\right)^9$. [3 marks]
- (b) Hence, or otherwise, find the term independent of x . [2 marks]
4. Given that $z_1 = 2 + 3i$ and $z_2 = p - 6i$, $p \in \mathbb{R}$, find:
- (a) $z_1 \bar{z}_2$; [2 marks]
- (b) the value of p such that $z_1 \bar{z}_2$ is a real number. [1 mark]
5. Use the Euclidean algorithm to find integers a and b such that $306a + 119b = 17$. [4 marks]

6. On a suitable domain, a curve is defined parametrically by $x = t^2 + 1$ and $y = \ln(3t + 2)$.

Find the equation of the tangent to the curve where $t = -\frac{1}{3}$. [5 marks]

7. Matrices C and D are given by:

$$C = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 1 & 2 \\ k+3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \text{ where } k \in \mathbb{R}.$$

(a) Obtain $2C' - D$ where C' is the transpose of C . [2 marks]

(b) (i) Find and simplify an expression for the determinant of D . [2 marks]

(ii) State the value of k such that D^{-1} does not exist. [1 mark]

8. Using the substitution $u = \sin \theta$, or otherwise, evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2 \sin^4 \theta \cos \theta \, d\theta. \quad [4 \text{ marks}]$$

9. Prove directly that:

(a) the sum of any three consecutive integers is divisible by 3; [2 marks]

(b) any odd integer can be expressed as the sum of two consecutive integers. [1 mark]

10. Given $z = x + iy$, sketch the locus in the complex plane given by $|z| = |z - 2 + 2i|$. [3 marks]

11. (a) Obtain the matrix, A , associated with an anticlockwise rotation of $\frac{\pi}{3}$ radians about the origin. [1 mark]
- (b) Find the matrix, B , associated with a reflection in the x -axis. [1 mark]
- (c) Hence obtain the matrix, P , associated with an anticlockwise rotation of $\frac{\pi}{3}$ radians about the origin followed by reflection in the x -axis, expressing your answer using exact values. [2 marks]
- (d) Explain why matrix P is not associated with rotation about the origin. [1 mark]

12. Prove by induction that, for all positive integers n ,

$$\sum_{r=1}^n 3^{r-1} = \frac{1}{2}(3^n - 1). \quad [5 \text{ marks}]$$

*13. Refer to the diagram for Question 13. An engineer has designed a lifting device. The handle turns a screw which shortens the horizontal length and increases the vertical height.

The device is modelled by a rhombus, with each side 25 cm, as shown in the diagram.

The horizontal length is x cm, and the vertical height is h cm as shown.

(a) Show that $h = \sqrt{2500 - x^2}$. [1 mark]

(b) The horizontal length decreases at a rate of 0.3 cm per second as the handle is turned. Find the rate of change of the vertical height when $x = 30$. [5 marks]

14. A geometric sequence has first term 80 and common ratio $\frac{1}{3}$.

(a) For this sequence, calculate:

(i) the 7th term; [2 marks]

(ii) the sum to infinity of the associated geometric series. [2 marks]

The first term of this geometric sequence is equal to the first term of an arithmetic sequence.

The sum of the first five terms of this arithmetic sequence is 240.

(b) (i) Find the common difference of this sequence. [2 marks]

(ii) Write down and simplify an expression for the n th term. [1 mark]

Let S_n represent the sum of the first n terms of this arithmetic sequence.

(c) Find the values of n for which $S_n = 144$. [3 marks]

15. (a) Use integration by parts to find $\int x \sin 3x \, dx$. [3 marks]

(b) Hence find the particular solution of

$$\frac{dy}{dx} - \frac{2}{x}y = x^3 \sin 3x, \quad x \neq 0$$

given that $x = \pi$ when $y = 0$.

Express your answer in the form $y = f(x)$. [7 marks]

16. Planes π_1 , π_2 and π_3 have equations:

$$\pi_1: \quad x - 2y + z = -4$$

$$\pi_2: \quad 3x - 5y - 2z = 1$$

$$\pi_3: \quad -7x + 11y + az = -11$$

where $a \in \mathbb{R}$.

- (a) Use Gaussian elimination to find the value of a such that the intersection of the planes π_1 , π_2 and π_3 is a line. **[4 marks]**
- (b) Find the equation of the line of intersection of the planes when a takes this value. **[2 marks]**

The plane π_4 has equation $-9x + 15y + 6z = 20$.

- (c) Find the acute angle between π_1 and π_4 . **[3 marks]**
- (d) Describe the geometrical relationship between π_2 and π_4 .
Justify your answer. **[1 mark]**

17. (a) Given $f(x) = e^{2x}$, obtain the Maclaurin expansion for $f(x)$ up to, and including, the term in x^3 . **[2 marks]**

(b) On a suitable domain, let $g(x) = \tan x$.

(i) Show that the third derivative of $g(x)$ is given by

$$g'''(x) = 2\sec^4 x + 4\tan^2 x \sec^2 x. \quad \mathbf{[3 \text{ marks}]}$$

(ii) Hence obtain the Maclaurin expansion for $g(x)$ up to and including the term in x^3 . **[2 marks]**

(c) Hence, or otherwise, obtain the Maclaurin expansion for $e^{2x} \tan x$ up to, and including, the term in x^3 . **[2 marks]**

(d) Write down the first three non-zero terms in the Maclaurin expansion for $2e^{2x} \tan x + e^{2x} \sec^2 x$. **[1 mark]**

[END OF QUESTION PAPER]

