National
Qualifications
2021
RESOURCE

2021 Mathematics of Mechanics

Advanced Higher

Finalised Marking Instructions

General marking principles for Advanced Higher Mathematics of Mechanics

Always apply these general principles. Use them in conjunction with the detailed marking instructions, which identify the key features required in candidates' responses.

The marking instructions for each question are generally in two sections:

generic scheme — this indicates why each mark is awarded
illustrative scheme — this covers methods which are commonly seen throughout the marking

In general, you should use the illustrative scheme. Only use the generic scheme where a candidate has used a method not covered in the illustrative scheme.

(a) Always use positive marking. This means candidates accumulate marks for the demonstration of relevant skills, knowledge and understanding; marks are not deducted for errors or omissions.

(b) If you are uncertain how to assess a specific candidate response because it is not covered by the general marking principles or the detailed marking instructions, you must seek guidance from your team leader.

(c) One mark is available for each •. There are no half marks.

(d) If a candidate's response contains an error, all working subsequent to this error must still be marked. Only award marks if the level of difficulty in their working is similar to the level of difficulty in the illustrative scheme.

(e) Only award full marks where the solution contains appropriate working. A correct answer with no working receives no mark, unless specifically mentioned in the marking instructions.

(f) Candidates may use any mathematically correct method to answer questions, except in cases where a particular method is specified or excluded.

(g) If an error is trivial, casual or insignificant, for example $6 \times 6 = 12$, candidates lose the opportunity to gain a mark, except for instances such as the second example in point (h) below.

(h) If a candidate makes a transcription error (question paper to script or within script), they lose the opportunity to gain the next process mark, for example

This is a transcription error and so the mark is not awarded.

This is no longer a solution of a quadratic equation, so the mark is not awarded.

$$x^2 + 5x + 7 = 9x + 4$$
$$x - 4x + 3 = 0$$
$$x = 1$$

The following example is an exception to the above

This error is not treated as a transcription error, as the candidate deals with the intended quadratic equation. The candidate has been given the benefit of the doubt and all marks awarded.

$$x^2 + 5x + 7 = 9x + 4$$
$$x - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 1 \text{ or } 3$$

(i) Horizontal/vertical marking

If a question results in two pairs of solutions, apply the following technique, but only if indicated in the detailed marking instructions for the question.

Example:

	$\bullet^5$	$\bullet^6$
$\bullet^5$	$x = 2$	$x = -4$
$\bullet^6$	$y = 5$	$y = -7$

Horizontal: $\bullet^5$ $x = 2$ and $x = -4$ Vertical: $\bullet^5$ $x = 2$ and $y = 5$
$\bullet^6$ $y = 5$ and $y = -7$ $\bullet^6$ $x = -4$ and $y = -7$

You must choose whichever method benefits the candidate, **not** a combination of both.

(j) In final answers, candidates should simplify numerical values as far as possible unless specifically mentioned in the detailed marking instruction. For example

$\frac{15}{12}$ must be simplified to $\frac{5}{4}$ or $1\frac{1}{4}$ $\frac{43}{1}$ must be simplified to 43

$\frac{15}{0\cdot3}$ must be simplified to 50 $\frac{4/5}{3}$ must be simplified to $\frac{4}{15}$

$\sqrt{64}$ must be simplified to 8*

*The square root of perfect squares up to and including 100 must be known.

(k) Commonly Observed Responses (COR) are shown in the marking instructions to help mark common and/or non-routine solutions. CORs may also be used as a guide when marking similar non-routine candidate responses.

(l) Do not penalise candidates for any of the following, unless specifically mentioned in the detailed marking instructions:

- working subsequent to a correct answer
- correct working in the wrong part of a question
- legitimate variations in numerical answers/algebraic expressions, for example angles in degrees rounded to nearest degree
- omission of units
- bad form (bad form only becomes bad form if subsequent working is correct), for example

  $\left(x^3 + 2x^2 + 3x + 2\right)\left(2x + 1\right)$ written as
  $\left(x^3 + 2x^2 + 3x + 2\right) \times 2x + 1$
  $= 2x^4 + 5x^3 + 8x^2 + 7x + 2$
  gains full credit

- repeated error within a question, but not between questions or papers

(m) In any 'Show that…' question, where candidates have to arrive at a required result, the last mark is not awarded as a follow-through from a previous error, unless specified in the detailed marking instructions.

(n) You must check all working carefully, even where a fundamental misunderstanding is apparent early in a candidate's response. You may still be able to award marks later in the question so you must refer continually to the marking instructions. The appearance of the correct answer does not necessarily indicate that you can award all the available marks to a candidate.

(o) You should mark legible scored-out working that has not been replaced. However, if the scored-out working has been replaced, you must only mark the replacement working.

(p) If candidates make multiple attempts using the same strategy and do not identify their final answer, mark all attempts and award the lowest mark. If candidates try different valid strategies, apply the above rule to attempts within each strategy and then award the highest mark.

For example:

Strategy 1 attempt 1 is worth 3 marks.	Strategy 2 attempt 1 is worth 1 mark.
Strategy 1 attempt 2 is worth 4 marks.	Strategy 2 attempt 2 is worth 5 marks.
From the attempts using strategy 1, the resultant mark would be 3.	From the attempts using strategy 2, the resultant mark would be 1.

In this case, award 3 marks.

Section 1

Detailed Marking Instructions for each question

Question			Generic scheme	Illustrative scheme	Max mark
1.			•[1] determine initial momentum of the system •[2] apply principle of conservation of momentum •[3] calculate value of m	•[1] $0{\cdot}75m - 0{\cdot}2 \times 400$ •[2] $0\cdot 75m - 0\cdot 2\times 400 = 0\cdot 37\left(m+400\right)$ •[3] $m = 600$	**3**

Question			Generic scheme	Illustrative scheme	Max mark
2.			•[1] correct use of product rule with one term correct •[2] complete differentiation •[3] evaluate	•[1] $3e^{3x}\sin 2x + \ldots$ or $\ldots + 2e^{3x}\cos 2x$ •[2] $3e^{3x}\sin 2x + 2e^{3x}\cos 2x$ •[3] $3e^{\frac{3\pi}{4}}$ or 31·7	3

Question			Generic scheme	Illustrative scheme	Max mark
3.	(a)		•[1] resolve forces vertically •[2] calculate trig ratio •[3] substitute and rearrange	•[1] $2T\sin\theta = mg$ •[2] $\sin\theta = \frac{\sqrt{5}}{3}$ •[3] $\frac{3mg}{2\sqrt{5}}$ or $0{\cdot}671mg$	**3**
	(b)		•[4] comment about tension increasing	•[4] eg as θ decreases, tension increases	**1**

Question			Generic scheme	Illustrative scheme	Max mark
4.	(a)		•¹ find expression for velocity of either P or Q •² find second expression for velocity •³ equate expressions for velocity and start to solve •⁴ find value of t	•¹ t^2+t+2 or $3t+5$ •² $3t+5$ or t^2+t+2 •³ $t^2-2t-3=0$ •⁴ $t=3$ (seconds)	**4**
	(b)		•⁵ find displacement of one particle •⁶ determine other displacement and find distance	•⁵ $s_P(3)=\frac{41}{2}$ or $s_Q(3)=\frac{57}{2}$ •⁶ $s_Q(3)=\frac{57}{2}$ $s_P(3)=\frac{41}{2}$ Distance $= 35\cdot1\ldots$(metres)	**2**

Question			Generic scheme	Illustrative scheme	Max mark
5.			•[1] differentiate and begin to substitute	•[1] $\frac{du}{dx} = -6x$ and evidence of substitution	4
			•[2] rewrite integral	•[2] $-\frac{1}{2}\int u^{\frac{1}{2}}\,du$	
			•[3] include limits	•[3] $\ldots\int_4^1 \ldots du$	
			•[4] integrate and evaluate	•[4] $\left[-\frac{u^{\frac{3}{2}}}{3}\right]_4^1$ and $\frac{7}{3}$	

Question			Generic scheme	Illustrative scheme	Max mark
6.	(a)		•¹ consider particle travelling vertically upwards	•¹ $v = u - 3g$	**3**
			•² consider particle falling	•² $v = 3g$	
			•³ find value for u	•³ $6g$ or $58 \cdot 8\left(\text{ms}^{-1}\right)$	
	(b)		•⁴ find displacement of one particle	•⁴ $13 \cdot 5g$ or $4 \cdot 5g$	**2**
			•⁵ find total distance	•⁵ $18g$ or $176 \cdot 4$ (m)	

Question			Generic scheme	Illustrative scheme	Max mark
7.	(a)	(i)	•[1] calculate normal reaction •[2] calculate frictional force •[3] calculate work done against friction	•[1] $R=\frac{2}{5}g$ •[2] $F=\frac{1}{4}g$ •[3] $\frac{7}{8}g$ or $8\cdot575(\text{J})$	3
		(ii)	**Method 1** •[4] consider total energy at points A and B •[5] calculate vertical height between A and B •[6] calculate speed **Method 2** •[4] resolve parallel to slope •[5] calculate acceleration •[6] calculate speed	**Method 1** •[4] $E_\text{A}=\frac{1}{2}\left(\frac{1}{2}\right)u^2,\ E_\text{B}=\frac{1}{2}g\left(\frac{7}{2}\sin\theta\right)$ •[5] $h=\frac{21}{10}$ •[6] $\sqrt{\frac{77}{10}g}$ or $8\cdot69(\text{ms}^{-1})$ **Method 2** •[4] $0\cdot5a=-0\cdot5g\sin\theta-\mu R$ •[5] $a=-10\cdot78$ •[6] $\sqrt{\frac{77}{10}g}$ or $8\cdot69(\text{ms}^{-1})$	3
	(b)		•[7] apply work-energy principle •[8] calculate speed **Alternative:** •[7] calculate acceleration down the slope •[8] calculate speed	•[7] $\frac{7}{8}g=\frac{21}{20}g-\frac{1}{4}u^2$ •[8] $\sqrt{\frac{7}{10}g}$ or $2\cdot62(\text{ms}^{-1})$ **Alternative:** •[7] $a=\frac{1}{10}g$ •[8] $\sqrt{\frac{7}{10}g}$ or $2\cdot62(\text{ms}^{-1})$	2

Question			Generic scheme	Illustrative scheme	Max mark
8.	(a)		•[1] use product rule with 1 term correct •[2] complete differentiation •[3] evaluate derivatives of x and y at $t=\frac{\pi}{6}$ •[4] calculate angle and state direction relative to x-axis	•[1] $1.\sin 2t + \ldots$ or $t.2\cos 2t + \ldots$ •[2] $\frac{dx}{dt} = \sin 2t + 2t\cos 2t$ **and** $\frac{dy}{dt} = -3\sin 3t$ •[3] $\frac{dx}{dt} = \frac{\pi}{6} + \frac{\sqrt{3}}{2}$, $\frac{dy}{dt} = -3$ •[4] $65{\cdot}1°$ or $1{\cdot}14$ radians below the x-axis	**4**
	(b)		•[5] calculate speed	•[5] $3{\cdot}31\ldots$ (ms^{-1})	**1**

Question			Generic scheme	Illustrative scheme	Max mark
9.	(a)	(i)	•[1] state position vector of A at time t •[2] state position vector of B at time t •[3] find the relative position vector •[4] find expression for the square of the magnitude •[5] differentiate and solve for t	•[1] $\begin{pmatrix} 14t \\ 6t \end{pmatrix}$ •[2] $\begin{pmatrix} 10t \\ 6-4t \end{pmatrix}$ •[3] $\begin{pmatrix} 4t \\ 10t-6 \end{pmatrix}$ •[4] $116t^2-120t+36$ •[5] $0\cdot52$ hours (or 31 minutes)	**5**
		(ii)	•[6] find distance of closest approach	•[6] 2·2 (km)	**1**
	(b)		•[7] interpret condition for A to be east of B •[8] calculate time interval	•[7] $10t-6=0$ or equivalent •[8] 5 minutes	**2**

Question			Generic scheme	Illustrative scheme	Max mark
10.			•¹ identify integral form of integrating factor •² determine integrating factor •³ state integral equation •⁴ integrate •⁵ evaluate constant •⁶ rewrite in required form	•¹ $I.F = e^{\int -2x\,dx}$ •² e^{-x^2} or $\frac{1}{e^{x^2}}$ •³ $\frac{y}{e^{x^2}} = \int \frac{1}{(x+1)^2}dx$ •⁴ $\frac{y}{e^{x^2}} = -\frac{1}{x+1} + c$ •⁵ $c = 6$ •⁶ $y = 6e^{x^2} - \frac{e^{x^2}}{x+1}$	6

Question			Generic scheme	Illustrative scheme	Max mark
11.			•¹ calculate radius •² resolve forces in vertical equilibrium •³ calculate T in terms of m •⁴ use Newton's Second Law horizontally •⁵ substitute values •⁶ calculate ω	•¹ 12 •² $4T\sin\alpha - T\sin\beta = mg$ •³ $T = \dfrac{65mg}{131}$ •⁴ $4T\cos\alpha + T\cos\beta = mr\omega^2$ •⁵ $\dfrac{16}{5}\left(\dfrac{65g}{131}\right) + \dfrac{12}{13}\left(\dfrac{65g}{131}\right) = \omega^2 0\cdot12$ •⁶ $12\cdot9\left(\text{rads}^{-1}\right)$	**6**

Question			Generic scheme	Illustrative scheme	Max mark
12.			•¹ differentiate product term or apply chain rule	•¹ $\ldots 2y + 2x\frac{dy}{dx}\ldots$ or $-2y\frac{dy}{dx}$	**5**
			•² complete differentiation	•² $6x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$	
			•³ differentiate either product	•³ $\ldots 2\frac{dy}{dx} + 2x\frac{d^2y}{dx^2}\ldots$ or $\ldots -2\left(\frac{dy}{dx}\right)^2 - 2y\frac{d^2y}{dx^2}\ldots$	
			•⁴ complete differentiation	•⁴ $6 + 2\frac{dy}{dx} + 2\frac{dy}{dx} + 2x\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 - 2y\frac{d^2y}{dx^2} = 0$	
			•⁵ communicate and complete	•⁵ either $6 + 2x\frac{d^2y}{dx^2} - 2y\frac{d^2y}{dx^2} = 0$ or $\frac{dy}{dx} = 0$ and $3 + (x - y)\frac{d^2y}{dx^2} = 0$	

Question			Generic scheme	Illustrative scheme	Max mark
13.	(a)		•¹ use Newton's Second Law	•¹ $mg - mkv^2 = ma$	**6**
			•² separate variables	•² $\int dx = \int \frac{v}{g - kv^2} dv$	
			•³ integrate	•³ $x + c = -\frac{1}{2k} \ln\left(g - kv^2\right)$	
			•⁴ calculate constant of integration	•⁴ $c = -\frac{1}{2k} \ln g$	
			•⁵ substitute constant and form exponential equation	•⁵ $e^{2kx} = \frac{g}{g - kv^2}$	
			•⁶ complete proof	•⁶ $... \Rightarrow v = \sqrt{\frac{g\left(1 - e^{-2kx}\right)}{k}}$	
	(b)		•⁷ apply condition for terminal velocity	•⁷ $mg - mkv^2 = 0$	**2**
			•⁸ evaluate constant	•⁸ $k = 0 \cdot 00348\ldots$	
	(c)		•⁹ substitute values and start to evaluate	•⁹ $52 \cdot 47 = \sqrt{\frac{g\left(1 - e^{-2 \times 0 \cdot 00349 x}\right)}{0 \cdot 00349}}$	**2**
			•¹⁰ evaluate distance	•¹⁰ $561 \cdot 4$	

Question			Generic scheme	Illustrative scheme	Max mark
14.	(a)		•¹ find expression for horizontal displacement of P •² find expression for vertical displacement of P •³ determine expressions for horizontal and vertical displacement of Q •⁴ equate horizontal and vertical components •⁵ complete proof	•¹ $u_{\text{P}}t\cos\alpha$ •² $\left(u_{\text{P}}t\sin\alpha-\frac{1}{2}gt^2\right)+h$ •³ $d-u_{\text{Q}}t\cos\beta$ $u_{\text{Q}}t\sin\beta-\frac{1}{2}gt^2$ •⁴ $u_{\text{P}}t\cos\alpha=d-u_{\text{Q}}t\cos\beta$ $u_{\text{P}}t\sin\alpha-\frac{1}{2}gt^2+h=u_{\text{Q}}t\sin\beta-\frac{1}{2}gt^2$ •⁵ $d=t\left(u_{\text{P}}\cos\alpha+u_{\text{Q}}\cos\beta\right)$ $h=t\left(u_{\text{Q}}\sin\beta-u_{\text{P}}\sin\alpha\right)$	5
	(b)		•⁶ form equation •⁷ substitute and complete	•⁶ $u_{\text{P}}\cos\alpha+u_{\text{Q}}\cos\beta=3\left(u_{\text{Q}}\sin\beta-u_{\text{P}}\sin\alpha\right)$ •⁷ 1·26	2

ALTERNATIVE SOLUTION					
Question			**Generic scheme**	**Illustrative scheme**	**Max mark**
14.	(a)		•1 use equation of motion to give vertical displacement of P •2 use equation of motion to give vertical displacement of Q •3 combine equations from •1 and •2 to eliminate y and give expression for h •4 state horizontal displacements of both P and Q •5 find expression for total displacement	•1 $s = ut + \frac{1}{2}at^2$ $y - h = u_P \sin\alpha \times t - \frac{g}{2}t^2$ •2 $y = u_Q \sin\beta \times t - \frac{g}{2}t^2$ •3 $u_Q \sin\beta \times t - \frac{g}{2}t^2 - h = u_P \sin\alpha \times t - \frac{g}{2}t^2$ $h = t\left(u_Q \sin\beta - u_P \sin\alpha\right)$ •4 $u_P t \cos\alpha \qquad d - u_Q t \cos\beta$ •5 $u_P t \cos\alpha = d - u_Q t \cos\beta$ $\Rightarrow d = t(u_P \cos\alpha + u_Q \cos\beta)$	**5**

Section 2

Part A

Question			Generic scheme	Illustrative scheme	Max mark
15.	(a)	(i)	•[1] calculate period	•[1] 6	**1**
		(ii)	•[2] find α	•[2] $\frac{\pi}{6}$	**1**
	(b)		•[3] substitute amplitude and phase angle •[4] calculate time	•[3] $8 = 8\sin\left(\frac{\pi}{3}t + \frac{\pi}{6}\right)$ •[4] 1	**2**

Question			Generic scheme	Illustrative scheme	Max mark
16.	(a)		•[1] apply Hooke's law with substitution •[2] find equilibrium extension and complete	•[1] $\frac{4mgx_e}{l} = mg$ •[2] $x_e = \frac{l}{4} \Rightarrow \text{AB} = l + \frac{l}{4} = \frac{5l}{4}$	**2**
	(b)	(i)	•[3] form equation and substitute for tension •[4] substitute values and complete	•[3] $mg - \frac{\lambda(x + x_e)}{l} = m\ddot{x}$ •[4] $mg - \frac{4mg\left(x + \frac{l}{4}\right)}{l} = m\ddot{x}$ $\Rightarrow \ddot{x} = -\frac{4g}{l}x$	**2**
		(ii)	•[5] interpret equation •[6] calculate maximum acceleration	•[5] $\omega^2 = \frac{4g}{l}$ •[6] $\frac{4g}{5} = 7\cdot84\ \text{ms}^{-2}$	**2**

Part B

Question			Generic Scheme	Illustrative Scheme	Max Mark
17.	(a)		•[1] consider moments about the centre •[2] equate to turning effect and solve for x	•[1] $80gx - 2\times16g - 0.5\times16g$ •[2] 0.75	**2**
	(b)		•[3] consider moments in the anticlockwise direction •[4] equate to turning effect and solve for y	•[3] $2\times16g + 0.5\times16g - 80gy$ •[4] 0.25	**2**

Question			Generic scheme	Illustrative scheme	Max mark
18.	(a)		•[1] work out area of semicircle and triangle •[2] work out positions of COM for each •[3] take moments vertically by equating with centre of mass of earring •[4] calculate distance	•[1] $A_{\text{PQR}} = 180$ $A_{SC} = 50\pi$ •[2] Triangle: 6 Semicircle: $\dfrac{40}{3\pi}$ •[3] $(180 + 50\pi)mg\overline{y}$ $= 180mg \times 6 - 50\pi mg \times \dfrac{40}{3\pi}$ •[4] 1·23 (mm)	**4**
	(b)		•[5] interpret new situation •[6] calculate the distance	•[5] $\ldots - 25\pi mgy = 0$ •[6] 5·26 (mm)	**2**

[END OF MARKING INSTRUCTIONS]